Macroscopic Theories of Matter and Fields: A Thermodynamic Approach

Edited by L. I. SEDOV

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0 PREFACE

Academician *L. I. Sedov*

Institute of Mechanics, M. V. Lomonosov Moscow State University

For a scientific description of mechanical and, in general, physical objects, media, fields, and phenomena one must employ theoretical models. Mathematical modeling of physical space-time lies at the foundation of any theoretical interpretation of the world around us.

At present the main distinctive feature of the common idea of physical space-time is that of a four-dimensional continuum of points, each of which is given by a set of four numbers, the coordinates, with one of these numbers being of a temporal nature, which is reflected in the geometric properties of the four-dimensional continuum. There are different ways of giving a concrete physico-geometric interpretation of the mathematical model of space-time. In Newtonian mechanics, as we know, a three-dimensional Euclidean space is used as the model of space, while time is considered a scalar. Special relativity postulates that a pseudo-Euclidean space is the model for the four-dimensional physical space-time. Finally, in general relativity and its many generalizations we assume that the four-dimensional space-time is Riemannian, while locally it is pseudo-Euclidean (the case in special relativity). There are also other interpretations of models of space-time.

At present many scientific centers are further developing the ideas of general relativity and its applications to specific physical and astrophysical problems. At the same time in the formulation of various problems involving second-order partial differential equations reflecting the dynamics of a problem there is an urgent need to rationalize the theory of additional conditions (for one, boundary conditions) closely linked to the conditions at strong discontinuities that may appear inside four-dimensional volumes where the sought functions are continuous.

The problems of individualization of points in a gravitational field, energy, internal interactions in a gravitational field, and a global proper time, connected with the choice of reference frames, have been largely solved. However, the final incorporation of theoretically and physically correct notions and model interpretations must be linked to special observations of natural phenomena and to laboratory experiments. In other words, the basic idea of the covariance of physical laws still requires considerable explanation. This is especially true when we deal with such basic characteristics as mass and energy, since the main physical laws are formulated only for individualized volumes of various fields and parts of matter and this requires the introduction of special reference frames so as to individualize the various objects.

The law of motion of a continuous medium can be found theoretically via the dynamical equations of the specific problem or experimentally by measuring in the observer's reference frame the signals that bring the data on the events in the comoving reference frame. The second way is not always possible and, besides, involves distortions introduced, first, by the peculiarities of and the processes in the intermediate media, where the signals propagate, and, second, by the differences in the mechanical and, in general, physical states of the observer and the moving medium. On the one hand, these are the phenomena associated with scattering, absorption, or distortion of electromagnetic or acoustic perturbations in the intermediate medium, all sorts of "noise", and, on the other, phenomena similar to the Doppler effect and especially all sorts of relativistic effects; also phenomena associated with the differences in the electromagnetic characteristics and in the geometric dimensions and time intervals that exist in the observer's reference frame and the comoving frame.

Obviously, the examples we have just cited are strongly correlated with the choice of the observer's reference frame and reflect not only the properties and the various phenomena in the moving reference frame but the phenomena in the observer's reference frame and the state of the observer, which essentially is unimportant from the viewpoint of the physical laws at work in the moving reference frame. In this connection we can make the general statement that it is expedient to carry out experiments and establish the main physical relationships by measurements and by building theoretical models directly in the comoving reference frame either theoretically or experimentally. After this we can transform the results and formulate them in the observer's reference frame. Essentially, this is done in physical theories in the simplest cases, though sometimes not explicitly.

The results obtained in the comoving reference frame are independent of the state and motion of the observer and, therefore, have a simpler form than the results obtained after the above-mentioned transformations, which often involve complicated mathematical operations (these constitute the essence of navigational calculations). Roughly speaking, we can illustrate our point by comparing the Ptolemaic system, which puts the observer on Earth in the center of the reference frame, with the Copernican system, where the reference frame is comoving with the Sun, or by considering the motion of Jovian satellites in relation to Jupiter's center of mass, or by analyzing physical processes in atoms belonging to moving bodies when using reference frames connected with the nucleus of an atom. Other examples can also be cited.

A starting point of any experimental investigation is the perception of the meaning of the characteristic concepts that are being introduced, the listing of such concepts, and the establishment of relationships between these concepts, which either follow from their definitions or are given by the conditions of the experiment, or by the postulates in a theory. In other words, in the set of characteristic quantities we must fix explicitly an independent system of determining parameters and the characteristic quantities, functions, and laws that we set out to establish. These quantities, functions, and laws are obtained in the process of attaining the goals of the investigation via invented models that are put at the foundation of the study and are used in formulating over-simplified problems, so to say.

To build models and obtain closed systems of mathematical relationships we must use basic relationships, i.e. more or less universal relations which, on the one hand, are considered as laws that work in the different models and phenomena and, on the other, must serve as a testing ground for various hypotheses or laws that require verification when describing physical phenomena. For instance, assuming that space is Euclidean and time is absolute, after we have chosen the inertial system of coordinates and defined the concept of a material point, we can build an analytical mechanics of interacting discrete material points on the basis of Newton's second law of mechanics

$$m\mathbf{a} = \mathbf{F}, \tag{0.1}$$

where m is the mass of the point, $\mathbf{a}$ its acceleration with respect to an inertial system of coordinates, and $\mathbf{F}$ the acting force, which is either defined through (0.1) or fixed by postulates, i.e. laws that are verified in advance on the basis of experiments that must employ Eq. (0.1).

As is known, the basic relationship (0.1) in the general case or in important classes of phenomena can be substituted by other (equivalent) formulas, in particular, by generalizations of the virtual work principle and the basic variational equation. Note that in motions of a system at each point of which there are no "gyroscopic" forces the basic equation (0.1) is equivalent to the corresponding energy equation.

One of the most important alternative formulations of a system of equations of type (0.1) in analytical mechanics is one scalar equation for the virtual work principle or one scalar variational equation in the form of Lagrange.

At present mechanics and physics are merging. This is especially evident in the macroscopic theories of mechanics of continuous media. In many cases, along with purely mechanical characteristics, such as the displacement vectors of a point in the medium and the various derivatives of these vectors with respect to position and time, the thermodynamic and electromagnetic characteristics of the inner states of bodies and fields are also acquiring significance. An effective and plausible description of macroscopic relationships and models of continuous media with internal degrees of freedom proves impossible if we rely only on Eq. (0.1). This leads to the problem of finding basic relationships of a more general nature than (0.1), i.e. relationships that would contain as particular cases the basic laws whose fruitfulness has been demonstrated and proved in mechanics, physics, and chemistry in the study of material media and fields.

To build a model means to establish a complete set of relationships suitable for describing a broad class of motions and processes, which as a rule are inhomogeneous over a volume and transient. In such a general formulation, however, a preliminary experimental verification can only be of partial importance; experience, common sense and intuition can help in supplying the minimally necessary mathematical and physical assumptions. The authors of the various theories are encouraged and inspired in their creative work by the many examples in which laws first established for a restricted class of phenomena have proved to be applicable to a much broader class.

In view of the complexity of describing the properties and interaction mechanisms between neighboring particles of matter and fields, one must use in constructing new models those methods that are well verified by practice.

For a long time both mechanics and physics have used the following universal relationships: the law of energy conservation (which expresses the balance of all types of energies), the virtual work principle, and the Lagrange variational principle. The virtual work principle is usually formulated in terms of, or derived from, the equations of mechanics for purely mechanical phenomena. The Lagrange variational principle can also be derived for continuous mechanical and electromagnetic phenomena. When we wish to generalize these principles so as to include the possibility of internal degrees of freedom, we can take them as postulates by fixing the coefficients of the variations of the sought functions. Just as with the force in Newton's second law, the fixing of these coefficients amounts, on the whole, to choosing a specific model of the medium or field.

We note in passing that if the infinitesimal possible variations of the sought functions coincide with small increments of these functions in respective real processes, then it is easy to show that for a particle the energy equation, the equation representing the virtual work principle, and the basic variational equation coincide.

In cases where the real infinitesimal increments of the sought functions may be arbitrary (linear independence is sufficient), for processes in which the coefficients of the increments of these functions are independent of these increments it follows that the energy equation as well as the equation reflecting the virtual work principle and the variational equation lead directly (in the absence of additional constraints) to the equations of motion, state, and other equations associated with physico-chemical processes for internal varying characteristic parameters. But usually the equation reflecting the virtual work principle and the variational equation have a more general nature than the energy equation, which lacks the "gyroscopic terms" present in the virtual work equation and variational equation.

The papers presented in this collection use the basic variational equation to develop, starting from the laws of thermodynamics, a theory of building models of continuous media and fields. The equation is taken in the form

$$\delta \int_{V_4} \Lambda \, dV_4 + \delta W^* + \delta W = 0. \tag{0.2}$$

Here dV_4 is the element of an arbitrary four-dimensional volume V_4 of space-time.

In this equation we take as given the Lagrangian density Λ and the linear functional δW^* of the variations of the sought functions and of the derivatives of the variations with respect to position coordinates and time. This functional is represented by an integral over the volume V_4 and an integral over the three-dimensional surface $\Sigma - S_4$, where Σ is a surface bounding the volume V_4 and S_4 are two-sided surfaces on which the sought functions may undergo strong discontinuities. The functional δW is determined via Λ and δW^* and is an integral over the surface $\Sigma - S_4$. The arguments of the scalar Λ and of the coefficients of the variations in δW^* are expressed in terms of the sought functions and their derivatives (of various orders) with respect to the position coordinates and time.

In Newtonian mechanics, in most cases (but not always) we may put

$$\Lambda = \rho v^2/2 - \rho U, \tag{0.3}$$

where ρ is the mass density, v the particle velocity (i.e. the velocity of the particles of the medium being modeled) with respect to an inertial reference frame, and U the internal energy of the particle calculated per unit mass (a thermodynamic characteristic of the medium depending on the mechanical and thermodynamic parameters that characterize the state of infinitesimal particles of the medium). If one must include in the system an electromagnetic field, then additional terms appear on the right-hand side of (0.3).

The physical (thermodynamic) interpretation of δW^* in some cases reduces to the following. The integrand in δW^* is the total influx of external energy to an infinitesimal particle due to the variation of the determining parameters. This quantity incorporates the work of external forces, influxes of external heat $dQ^{(e)}$, and additional influxes of energy dQ^{**} of a nonthermal nature.

Since both V_4 and Σ are arbitrary, the process of finding the coefficients in the linear formula by the variations of the sought functions and their derivatives, which are present in the integrand of the surface integral for δW, amounts to establishing the equations of state for the model discussed. If we wish to find the equations of state when the entropy S or the temperature T are included in the determining parameters, we must use the second law of thermodynamics in the form

$$\int_{V_4} \rho T\, dS\, dV_4 = \int_{V_4} (dQ^{(e)} + dQ')\, dV_4/dV_3, \tag{0.4}$$

so as to exclude $dQ^{(e)}$ from δW^*. In Eq. (0.4), $dQ' \geqslant 0$ is the noncompensated heat. The fixing of dQ' is equivalent to introducing the laws of energy dissipation into the model discussed.

Hence, in introducing Λ and δW^* that fix the model we can rely on various physical and mathematical lines of reasoning related to the thermodynamic properties of the bodies in the model through the internal energy U and the dissipation laws, which determine dQ'. The introduction of dQ^{**} also depends on the allowed physical mechanisms of energy exchange between a given particle, the neighboring particles, and the external bodies and fields.

On the basis of physical considerations, Eq. (0.2) in its initial form (with $dQ^{(e)}$) can be postulated not only for processes that are continuous in space and time but for processes with strong discontinuities inside the medium being considered, discontinuities that occur on certain surfaces $S_{\pm}$. When such discontinuities do occur, we must introduce into δW^* an additional integral over $S_{\pm}$, one that reflects the physical fact that at discontinuities there are intense energy influxes along $S_{\pm}$.

Equation (0.2) is a natural generalization of the Lagrange variational principle of analytical mechanics. This generalization is based on the following important considerations.

(1) The variational equation (0.2) is extended to cover continuous media in which there are mechanical and physico-chemical processes connected with internal degrees of freedom and characterized by macroscopic parameters (generally of a nonmechanical nature).

(2) Equation (0.2) is used to describe irreversible phenomena (the Euler equations include not only the general conservation equations but also the transport equations of chemical reactions and phase

transitions and, if we allow for electrodynamic phenomena, the Maxwell equations).

(3) Equation (0.2) is taken for arbitrary volumes and for arbitrary continuous variations of the sought functions, including variations that do not vanish at the boundary Σ. This enables establishing the thermodynamic equations of state, which determine the characteristics of mechanical, chemical, and electrodynamic interactions in bodies and fields, and also finding the metric properties of space.

(4) Once we have fixed the Lagrangian Λ and the functional δW^* expressed in terms of external energy influxes to the particles, we can on the basis of the integral form of Eq. (0.2) establish the conditions on the surfaces of strong discontinuities, where the determining parameters in the medium experience jumps (these conditions may be used, for one, to formulate initial and boundary conditions).

(5) The integral equation (0.2) can serve as the basis of various approximate and numerical methods for solving specific problems, without resorting to differential equations with additional initial and boundary conditions at the discontinuities.

(6) Finally, Eq. (0.2) may be used to build special models of thin bodies such as plates, shells, or rods, of the motion of liquids in films or shallow channels (in thin layers of fluids), and in designing composite constructions. In such types of problems we can specify certain directions, e.g. the direction normal to the surface of a plate or a direction lying in the cross section of a rod, along which (if the plate is thin, say) the distribution of the state characteristics being studied may be presented in a simplified form by fixed functions. (In the case of a plate these functions are defined along the initial normal to the plate's surface and contain "internal" parameters, which may depend on the directions lying in the plane of the plate and must be varied and defined via Eq. (0.2) in the process of solving the problem.)

Application of a similar method in various modifications to the three-dimensional case may constitute a direct or indirect generalization of the Bubnov method.

Once we have substituted the sought functions in such a way, we can carry out certain integration processes in Eq. (0.2), after which the resulting variational equation reduces the problem to a more complex model with internal degrees of freedom but of lower dimension; e.g. for plates and shells the model is two-dimensional, while for rods it is one-dimensional.

In connection with the basic equation (0.2) the following remarks are in order.

Fixing the set of Euler equations that contain the momentum equation, the moment of momentum equation, the energy equation, the equation for entropy production, the transport equations, and the like (which may be interrelated) does not define the Lagrangian density Λ. Obviously, by adding to Λ nonzero "divergence" terms we do not change the Euler equations but do change the expression for δW, which influences the state equations. In this way, fixing a complete set of Euler equations, i.e. all equations valid in different processes, does not fix the equations of state. The authors of some textbooks and monographs assert that if we know the complete set of differential equations describing phenomena within a certain model, then we can find the conditions at strong discontinuities. But this is untrue. In the first place, discontinuous motion, generally speaking, cannot be considered as a limit of continuous motions within the same model and, in the second place, in deriving the conditions at strong discontinuities we are forced to use integral relationships, which for a given set of differential equations cannot be defined unambiguously. For continuous motions, a given set of differential equations has corresponding to it a variety of integral relationships all equivalent to this set. By contrast, for discontinuous motions each set of integral relationships gives different contradicting conditions. It is obvious then that to establish the conditions at strong discontinuities we must postulate as an experimentally established law a corresponding set of integral relationships that hold for both continuous processes and processes with strong discontinuities.

Experience shows that the integral relationship that can be applied not only to continuous motions but to discontinuous motions as well is the energy conservation law in integral form. On the other hand, the heat influx equation in integral form or the entropy conservation equation for adiabatic processes in integral form leads to wrong conditions at strong discontinuities. In view of this it is clear that the variational equation (0.2) in integral form can be used for discontinuous processes and to derive the conditions at discontinuities, provided that the expressions for the Lagrangian density Λ and the functional δW^* are physically meaningful (i.e. agree with experience). For one, when deriving the conditions at discontinuities we must retain for δW^* all terms characterizing energy influxes and introduce new concentrated influxes over the discontinuity surfaces in a form directly included in the energy equation. The function Λ must also be fixed (this determines the divergence term) by employing physical considerations, which in the final analysis are justified by the correspondence between the given model and experiments in which strong discontinuities in the state characteristics are observed.

The present compilation includes works devoted to the building and study of models of various media and fields using the basic variational equation.

In the papers by L. I. Sedov, "A thermodynamic approach to the basic variational equation for building models of continuous media" and "Applying the basic variational equation for building models of matter and fields", the methods of building such models are defined more precisely. The author gives a thorough thermodynamic basis that allows for inclusion of the interaction of continuous media and fields in reversible and irreversible processes. He develops a theory of variations and establishes the interrelationships between these variations in the comoving reference frame and the observer's reference frame. Variations of special types are used to establish identities that follow from the scalar nature of the Lagrangian, and the various determining parameters of continuous media and fields that enter the Lagrangian are discussed. Then the author establishes general methods for obtaining the conditions at strong discontinuities and considers specific models of fluids, elastic bodies and also models describing the interaction of material media and electromagnetic fields.

The paper "Transition from relativistic to Newtonian mechanics in the presence of irreversible processes" by L. T. Chernyi develops a general method of obtaining Newtonian models of continuous media starting from appropriate relativistic models. The main idea is to write the relativistic equations in the proper (or rest) reference frames, which are defined for each point of the continuous medium and at each moment of time, and then to go over to a unique global inertial reference frame of the observer by a Galilean transformation. This reveals that the various macroscopic relativistic effects can be divided into two types. The first includes relativistic effects observable in rest frames, while the second incorporates relativistic effects due to the transition from rest reference frames to a unique reference frame of the observer via a Lorentz transformation. The effects of the second type are associated with a navigational problem, while relativistic effects of the first type manifest themselves only in the dependence of the Lagrangian and the functional δW^* on the determining parameters in the rest frames and can be interpreted within the framework of Newtonian mechanics. This method is examined on the model of an elastoplastic, viscous, heat-conducting continuous medium with additional internal degrees of freedom.

In his next paper, "Models of ferromagnetic continuous media with magnetic hysteresis", the author builds models of magnetizable, elastoplastic continuous media with magnetic hysteresis starting from the basic variational equation. An essentially new feature here is the introduction of the residual magnetization as an additional determining parameter, for which a new transport (kinetic) equation is found. The author postulates that an increment in the entropy of an elastoplastic ferromagnetic medium with hysteresis depends solely on the changes in the plastic deformations and the residual magnetization and does not depend on the rate at which these changes occur. This assumption is based on the irreversible nature of quasistatic equilibrium (i.e. infinitely slow) processes of plastic deformation and the remagnetization of the medium. The author establishes a differential equation for the residual magnetization that enables him to build a phenomenological theory of magnetic hysteresis. He also considers gyromagnetic phenomena, irreversible magnetostriction, the effect of internal stresses on magnetization in plastically deformed ferromagnets, and the various concrete models of magnetizable continuous media.

In the article by V. A. Zhelnorovich, "Magnetizable and polarizable media with microstructure", the author offers a set of equations for the models of elastic and liquid magnetizable and polarizable media with microstructure within the scope of the Newtonian mechanics of Cosserat continua. As important physical parameters of the medium related to the Cosserat continuum he uses the vector of internal rotations (the angular velocity vector of the orthonormalized n-hedrals of the Cosserat continuum) and the tensor of internal "turns" (the Ricci tensor for the system of n-hedrals of the Cosserat continuum). The use of these characteristics enables the author to describe media with internal angular momentum and moment stresses. In the models discussed the author considers magnetizable and polarizable media with relaxation of the Debye type and magnetizable media with internal angular momentum related to the magnetization of the medium, for which he introduces the vector of internal rotations. Within the model of magnetizable liquid the author solves problems related to the propagation of spin waves and magneto-acoustic waves, establishes the magnetic resonance frequencies in an ellipsoidal volume of the liquid, and obtains exact solutions of the type of Riemann waves. To build the various models he also solves the problems of the plane-parallel Couette flow and the cylindrical Poiseuille flow.

In his article "On exact solutions for interacting gravitational and electromagnetic fields", G. A. Alekseev uses the general ideas and methods of integrating nonlinear systems, which first emerged in the inverse scattering theory, to develop a method for building broad families of solutions of the soliton type for a set of equations that describe the nonlinear interaction of electromagnetic and gravitational fields in the framework of general relativity. These families of solutions, which contain an arbitrary but finite number of free parameters, depend on two coordinates (either time and one spatial coordinate or two spatial coordinates) and describe the propagation of plane or cylindrical electromagnetic and gravitational waves with arbitrary polarization and also the nonlinear interaction of static axisymmetric electrovacuum fields from different sources with a complex multipole structure. Since the method allows not only finding solutions with an arbitrary number of free parameters but using as the background solution any known exact solution (depending only on two coordinates, of course), it can be employed to build various concrete (exact) models for processes involving nonlinear interaction of gravitational and electromagnetic fields.

The next article, "Neutrino fields in general relativity", by N. R. Sibgatullin, considers the properties of interacting fundamental fields (gravitational, neutrino, and electromagnetic) in the framework of general relativity. The focus is on developing methods of obtaining exact solutions of the Maxwell and Weyl system of equations in the presence of additional symmetry of an Abelian group of motions G_2 on V_2. The author also studies the propagation of discontinuities of the first kind in a neutrino electrovacuum. For static axisymmetric fields as well as for plane waves, the system of equations for the neutrino electrovacuum reduces to a system of canonical equations, which include the known equations of Geroch, of Hauser and Ernst, and of Kinnersley and Chitre as particular cases in the absence of neutrino fields. The author studies the group properties of these canonical equations and builds new classes of exact particular solutions. He considers the collision of neutrino and gravitational fields, gives exact solutions related to a static neutrino vacuum, and builds a solution for a black hole in arbitrary neutrino fields. When there are no neutrino fields, the solutions found by the author include as a particular case the solutions of Sato and Tomimatzu, of Belinskii and Zakharov, and of Kinnersley, Chitre, Hoenselaers, Xanthopoulos, Hauser, Ernst, and others. He shows that a neutrino field leads to a new effect, rotation of the plane of polarization of discontinuities in the gravitational field. If, in addition, external electromagnetic fields come into play, then in the first approximation the action of the neutrino fields on the propagation of weak discontinuities can be described by fixing an effective electromagnetic field; as established earlier by Sibgatullin (see reference [13] in the literature cited in his article), in arbitrary external electromagnetic fields there is successive transmutation of gravitational and electromagnetic discontinuities into each other and rotation of the polarization vectors corresponding to these discontinuities.

The last article, "Tensor representation of spinor fields", by V. A. Zhelnorovich, presents methods that enable fully determining spinors in a three-dimensional Euclidean and a four-dimensional pseudo-Euclidean space in terms of certain tensors. The components of these tensors, by definition, are connected by simple algebraic (bilinear) equations. A one-to-one invariant relation between the spinor components and the tensor components enables representing the spinor equations in the form of equivalent tensor equations. Important classes of spinor equations used in field theory and quantum mechanics are given as examples. The developed tools are used in the mechanics of continuous media to formulate integrals of nonlinear partial differential equations describing relativistic models of liquids.

Thus, the reader can see that using the basic variational equation (0.2) and the physical meaning of its terms we can establish and then use the various links with thermodynamics, electrodynamics, and chemistry.

Moscow
November, 1982

L.I. Sedov

1

A Thermodynamic Approach to the Basic Variational Equation for Building Models of Continuous Media

Academician *L. I. Sedov*

Institute of Mechanics, M. V. Lomonosov Moscow State University

Theoretical and experimental studies in physics, and in general in the natural sciences, are based on the utilization of a number of concepts—the characteristics of objects and phenomena—which in many cases are presented as mathematical abstractions introduced through a system of axioms or formulas and various mathematical relations that enable the researcher, at least in principle, to find the characteristics by logically developed operations or by observation and measurement in experiments or by a combination of the two.

In other words, problems are formulated, described and solved by introducing models that must reflect the basic and important properties of substances and fields and a grasp of the processes in our environment and at the same time serve as a basis for understanding and solving many problems of technology and other aspects of everyday life.

A number of basic concepts are introduced axiomatically as primary and others as secondary, or derived, concepts expressed in certain ways through the primary [1-6].

Among such characteristic objects defined exactly in mathematics are the model concepts of space and time, the concept of mass as the measure of inertia or as a characteristic of the gravitational aspects of substance, and the like.

The basic concepts in mechanics are the ideas of geometric points of space, of time, and of reference frames, with whose help we introduce spatial and temporal coordinates.

Physical space and time are taken as a four-dimensional geometric manifold of points. In Newtonian mechanics this manifold forms the exactly defined and well-known Euclidean space while time is considered an absolute scalar measurable by clocks at each point in space with a result independent of the motion and position of the point and clock.

This model of space and time is well suited for describing many phenomena of practical importance and serves as a basis for other models of space and time.

For instance, in special relativity theory, which is an outgrowth of the theory of the electromagnetic field, three-dimensional space and one-dimensional time are taken as an integral whole, the four-dimensional pseudo-Euclidean space.

In general relativity, space and time are taken as a curved four-dimensional pseudo-Riemannian space, which is not given a priori, as is the case with the pseudo-Euclidean space in special relativity, but is determined by the phenomena in which matter and fields, including electromagnetic fields, take part.

The problem of finding physical models of space and time can by no means be considered closed. Further developments of these concepts are being discussed in works concerned with gnoseological problems, e.g. in the supergravity theories one must consider four-dimensional metric affinely-connected non-Riemannian spaces with torsion.

Nevertheless, each of the classical theories, such as Newtonian mechanics, special relativity, and general relativity, will always retain its importance for the cognition of nature with the proper limits. The same is true of other physical models of matter and fields, such as the material point model, the models of a rigid body and an ideal fluid, the models in elasticity theory, some models in plasticity theory and the Maxwell theory of electromagnetic fields—in general, of any model that is not only rational but has been well tested in experiments or important fields of application [17].

The essence of models of space and time is the same as of any other model objects and formulations of problems. For instance, in constructing rockets or television towers we can assume them to be bars or beams as a first approximation. The initially rough schemes become more refined and detailed to meet the requirements of technology and with the need for a fuller understanding. But in relation to models of space and time, this process has a more universal nature and is of basic importance to science.

It must be noted that the essence of Newtonian mechanics and the meaning of its laws become clearer after we have gone beyond its limits in practical applications and have to deal with special or general relativity. This is the case, for instance, when we wish to understand the fundamental aspects of inertia and inertial reference frames and the nature of gravity and inertial forces, or when we are interested in motion by inertia or the meaning of the invariant readings of the spring accelerometer, readings independent of the underlying theory or the reference frame used [23].

Perhaps even today in the everyday problems of physics the chief importance of general relativity is the greater insight that it gives into the essence and meaning of Newtonian mechanics. This idea should be the guiding principle in constructing new models.

No special understanding is needed to follow the guidelines just presented. Creative thinking that would generate new ideas and help obtain new results requires a profound understanding of the meaning of the modeling process. But strange as it may seem, the understanding of simple basic facts is hindered by a mass of prescriptions embedded in our thinking by an imperfect learning process.

The above-stated truths may seem trivial, but this is not so. A well-known professor once said at a conference, "You keep talking about models, but we are studying metals!"

It is characteristic of our times that the study of various phenomena is getting more and more complex and allows for important and subtle details that earlier were simply ignored. Versatile computers and measuring devices of astonishing accuracy facilitate finding answers to correctly posed questions. Hence, the focus of studies is shifting to a deeper theoretical investigation, one that involves modeling and correct formulation of problems [6, 17, 18, 22]. At times formulating suitable schemes and defining mathematical problems are among the main scientific results; they represent more than half the work of reaching the goals set in practice or theory.

Often the restricted applicability of new models narrows the very meaning of them as useful and effective tools, albeit crude, for studying various phenomena. The precision of these tools is dictated both by the requirements of practice and technology and by the goals set by fundamental studies of nature.

Summarizing, we may say that theoretical and practical construction of new models of matter and fields is a most essential problem with its own rich history. But experience has shown that there is a need to regulate the work in this field in accordance with the existing effective methods of interpreting these questions. This presupposes the use of modern achievements in physics and as a basis the use of the universal laws of physics, such as the first and second laws of thermodynamics, the covariance of physical laws in their mathematical formulation, the tested ways of accounting for the macroscopic irreversibility of phenomena, and the like.

The flood of works in which new phenomenological models are offered must be rationalized. This is most essential in generalizing the laws found theoretically or experimentally for simple particular cases to processes of a general nature.

One last remark is in order. The common concept of energy, forces, and interaction mechanisms inside matter and fields must be refined explicitly only after a sufficient understanding has been achieved. In some cases these refinements, in both the microscopic and the macroscopic approach, constitute the most important subject of current studies, since often these questions have no clear answers as yet.

In Newtonian mechanics the dynamical relationships of analytical mechanics and of the simplest models of continuous media are based on the laws of motion of Newton and additional laws for the forces acting in a particular system, laws that have been checked experimentally and agree with the Newtonian laws.

Let us take the fundamental equation for a material point or a body in translational motion

$$m\mathbf{a} = \mathbf{F} \tag{1.1}$$

where m is the mass, **a** the acceleration vector of the material point with respect to an inertial system of coordinates, and **F** the vector sum of all forces acting on the point.

Equation (1.1) is in a known manner generalized to the model of a system of material points with applied constraints.

Obviously, in applications the vector equation (1.1) and its generalizations are equivalent to a scalar equation for appropriate virtual displacements

$$(-md^2x_i/dt^2 + X_i)\,\delta x^i = 0, \tag{1.2}$$

where $x_i = x^i$ $(i = 1, 2, 3)$ are the Cartesian coordinates of the point, t the time, X_i the components of the force **F**, and δx^i the mentally introduced components of the vector $\delta\mathbf{r}$ of virtual displacements of the moving medium points. (Here and in what follows we assume summation over the same covariant and contravariant indices.)

Each component X_i can be expressed as a sum of a potential term and a residual term:

$$X_i = -\partial U/\partial x^i + X_i', \quad \delta U = (\partial U/\partial x^i)\,\delta x^i.$$

The scalar function U is known as the potential energy.

Next we denote the three-dimensional scalar $mv^2/2$ by T and call it the kinetic energy, with $\mathbf{v} = d\mathbf{r}/dt$. After certain transformations, Eq. (1.2) can be expressed thus:

$$\delta\,(T - U) - d\,(mv_i\delta x^i)/dt + X_i'\delta x^i = 0. \tag{1.3}$$

The equivalent equations (1.1)-(1.3) can be written for real displacements, when $\delta x^i = dx^i$, in the form

$$-d\,(T + U) + X_i'\,dx^i = 0. \tag{1.4}$$

By definition, Eq. (1.4) implies that the total mechanical energy $T + U$ is not conserved, and its increment $d\,(T + U)$ is equal to $X_i'\,dx^i$, which may be nonzero and for finite bodies can be interpreted in many cases as the increment of other forms of energy in balance with changes in the mechanical energy.

The generalization of the energy equation (1.4) to the equation of the first law of thermodynamics, which does not directly follow from (1.1), takes the form

$$-d(T + U) + dQ = 0. \tag{1.5}$$

Here dQ stands for the influx of energy, which in the Newtonian mechanics of a material point can be attributed to the elemental work of certain generalized forces $\mathbf{F}'$, while in the mechanics of finite bodies it is considered equivalent to fluxes of other types of energy, sometimes thermal, in balance with changes in the mechanical energy.

The first law of thermodynamics in the mechanics of a material point coincides with Eq. (1.5) if $dQ = 0$, i.e. in the absence of transformations of the mechanical energy of the point into other forms. But if such transformations do take place, Eq. (1.5) is the energy conservation law in thermodynamics, while in mechanics no such law exists for purely mechanical energy.

In the theory of continuous media, dQ may have a purely mechanical meaning, e.g. the work of surface forces at the boundary of a body or the work of external body forces inside the body. Another example is the work of the body or surface forces of a higher order when the internal interactions are characterized by tensors of rank greater than two (e.g. a force couple). In general, however, Eq. (1.5) is the first law of thermodynamics, where dQ is the increment of energies of various types that can be interpreted as thermal, chemical, or electromagnetic energy.

The scalar relationships

$$\begin{aligned} \delta T - d\,(mv_i\delta x^i)/dt &= -m\,(\mathbf{a}\cdot\delta\mathbf{r}), \\ -dT &= -m\,(\mathbf{a}\cdot d\mathbf{r}) \end{aligned} \qquad (1.6)$$

are simply the elemental work of the external inertial force over the virtual displacement δx^i or over the real displacement dx^i.

The scalar equation

$$\delta\,(T - U) - d\,(mv_i\delta x^i)/dt + \delta Q = 0 \qquad (1.7)$$

is a generalized form of Eq. (1.3), which after we have summed (integrated) over all the particles becomes valid for finite bodies and which can be thought of as the equation for the law of virtual conservation of energy—the basic postulate of physics corresponding to the first law of thermodynamics. Note that Eqs. (1.3) and (1.7) contain $T - U$, while Eqs. (1.4) and (1.5) contain $-(T + U)$.

Instead of the fundamental equation (1.1) or its generalizations we may start at the fundamental energy equation (1.3) or (1.7) or the equivalent integral relationships of the variational type. These questions are studied in detail in analytic mechanics.

In Newtonian mechanics the elemental work of an external body force of inertia over the virtual displacement $\delta\mathbf{r}$ of the center of a small individual particle of mass Δm is

$$-\Delta m\,(\mathbf{a}\cdot\delta\mathbf{r}) = -\Delta m\,((d^2\mathbf{r}/dt^2)\cdot\delta\mathbf{r}), \qquad (1.8)$$

where $\mathbf{a}$ is the acceleration vector with respect to any inertial reference frame, e.g. the local rest frame, $\mathbf{r}$ the radius vector of the particle, and $\delta\mathbf{r}$ the virtual change in $\mathbf{r}$.

The vectors $\mathbf{a}$, $\mathbf{r}$, and $\delta\mathbf{r}$ in Newtonian kinematics can be taken for any coordinate system. If we use a noninertial system of coordinates, which can be considered as a system of coordinates x^i in translatory motion, the following formulas are true: for velocity

$$\mathbf{v} = d\mathbf{r}/dt = \mathbf{v}_{(t)} + \mathbf{v}_{(r)}, \tag{1.9}$$

and for acceleration

$$\mathbf{a} = \mathbf{a}_{(t)} + \mathbf{a}_{(r)} + v^k_{(r)}\ (\partial\mathbf{v}_{(t)}/\partial x^k), \tag{1.10}$$

where $\mathbf{v}_{(t)}$ and $\mathbf{a}_{(t)}$ are the velocity and acceleration in any inertial reference frame of points in the translatory reference frame (coordinate system), and $\mathbf{v}_{(r)}$ and $\mathbf{a}_{(r)}$ are the three-dimensional velocity and acceleration taken in motion with respect to the translatory reference frame.

The third term on the right side of (1.10) represents a generalized Coriolis acceleration, which takes into account the stressed state of the translatory reference frame [27]. The vector formulas (1.9) and (1.10) can be written for each components in any system of coordinates.

We can write Eq. (1.8) at each point M in the following forms:

$$\begin{aligned}-\Delta m\,(\mathbf{a}\cdot\delta\mathbf{r}) &= \Delta m\,[\delta\,(v^2/2) - d\,(\mathbf{v}\cdot\delta\mathbf{r}/dt)]\\ &\left(\frac{\rho v^2}{2}\,dV_3\right) - \frac{\partial}{\partial t}\,[\rho\,(\mathbf{v}\cdot\delta\mathbf{r})\,dV_3] - \left[\frac{\partial}{\partial x^\alpha}\,\rho v^\alpha\,(\mathbf{v}\cdot\delta\mathbf{r})\right]dV_3\\ &= -\delta\left(\frac{\rho v^2}{2}\,dV_3\right) + \Delta m\left[\delta v^2 - d\left(\mathbf{v}\cdot\frac{\delta\mathbf{r}}{dt}\right)\right].\end{aligned} \tag{1.11}$$

Here dV_3 is the three-dimensional element of volume, and $\mathbf{v}$ is the velocity of the points of the medium with respect to the global inertial frame. The expression in the square brackets in the last line of (1.11) is zero for real motions, while the third term in the second line can be integrated by parts. This term is obtained from the equation of continuity in the form

$$\Delta m = \rho\,dV_3 = \text{const}, \quad \text{or} \quad \partial\rho/\partial t + \operatorname{div}\rho\mathbf{v} = 0. \tag{1.12}$$

In a local rest inertial reference tetrad, at each fixed point M we have $\mathbf{v} = \mathbf{v}^* = 0$. Hence

$$-\Delta m\ (\mathrm{a}\cdot\delta\mathbf{r}) = -\Delta m d\,(\mathbf{v}^*\cdot\delta\mathbf{r}^*/dt) \quad \text{and} \quad -\Delta m\,(\mathbf{a}\cdot d\mathbf{r}^*) = 0. \tag{1.13}$$

Thus, for real motions of small particles the energy equation (1.3) in rest reference frames does not contain the work of inertial forces and the kinetic energy, but the work of inertial forces over virtual displacements is generally nonzero.

In physics and chemistry equations of type (1.5) and (1.7) are considered fundamental in the studies of many phenomena, including those in which we may neglect the motions entirely or take into account only the motions due to changes in volume. In general, in both physics and chemistry when considering virtual states we must allow for the virtual work of inertial forces.

From the first law of thermodynamics for virtual or real processes it follows that the energy (including kinetic) of isolated individual systems, or systems that do not interact with the surroundings, is conserved if we allow for the work of inertial forces. When a system interacts with other systems, its energy may change only due to energy transfer between the various systems.

In order to write the general macroscopic equation of the first law of thermodynamics we must first discuss the nontrivial meaning of the following concepts:

(1) an individual physical object;

(2) the energy of physical objects, the various types of energies, and their distinctive features;

(3) the process of defining energy by different observers (both for the system as a whole and for its separate parts).

From the viewpoint of analytic mechanics, which deals with discrete systems of material points or with solid bodies that do not change, the concept (1) is quite clear. As to models of continuous media, the individuality of an isolated finite system as a whole is generally obvious, but the individuality of small parts of such a system is model dependent, i.e. depends on how we introduce the Lagrangian coordinates which specify the points in the medium.

In the final analysis, a macroscopically individual small particle of a continuous medium is simply a varying small volume of the substance; for this volume both the kinematic and physical states and the physical processes are governed by laws that by definition are true only for individual particles. In other words, individual particles are those whose states are described by laws for individual particles. By fixing the velocity field of individual points for each time t we define the Lagrangian coordinates that individualize the points in a medium.

To introduce a Lagrangian system of coordinates for a medium that is a mixture of various substances and fields for each of which the Lagrangian coordinates have already been chosen, we must rely on additional conditions that determine the characteristic physical properties of model individual particles for the entire continuum. When diffusion, chemical reactions, or transmutation of elementary and nuclear particles is present in the system, the common notion of an individual body, used in analytic mechanics becomes considerably more complex.

Indeed, if we consider the particles of matter of a small varying volume as a collection of the same atoms or molecules, then after a finite time interval has elapsed the individuality of the corresponding small particle in the continuous medium loses due to diffusion the usual meaning attributed to it by analytic mechanics. It is clear that even for a medium consisting of the same atoms the individuality of particles of a continuous medium for finite time intervals also depends on well-known mechanical assumptions introduced, implicitly or explicitly, in transferring from models of discrete media to models of continuously distributed continuous media.

Thus, the new concepts of individual particles of a medium in the mechanics of continuous media are closely linked with nontrivial axiomatic generalizations of microscopic concepts and laws (postulated instead of being introduced) to macroscopic concepts and laws formulated in the mechanics of discrete material points, which in turn constitute a far-reaching simplified ideal scheme of phenomena in the world around us.

In view of what has just been said we must also note that quantum mechanical effects in both internal and external interactions lead to a deeper level for the idea of individuality, e.g. when macroscopic phenomena can be described within the framework of classical mechanics (here the quantum effects manifest themselves through separately defined thermodynamic notions and through functional relationships introduced by quantum mechanics).

In connection with concept (2) we may ask: What is energy? First of all we must say that energy is the main model characteristic of physical objects (matter and fields) that can be taken either as a basic primary characteristic, similar to mass, in terms of which one can express other physical characteristics, or as a characteristic that can be expressed in terms of other characteristics taken as primary.

At this point it is important to note that in the case of any physical object we can always introduce and study in an infinitely small volume[1] its total energy and, in some cases, parts of this energy corresponding to the various types of energy. In every physical theory and for all physical objects, energy is the basic characteristic of a state and a process.

Of particular importance are such characteristics as total energy and its parts for individual objects considered over an infinitely small time interval or, if the meaning of an individualized object permits it, over finite time intervals.

The meaning and the very existence of the concept of energy for any physical object is derived from the first law of thermodynamics, which is a universal statement postulated for individual objects in all models of matter and fields.

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[1] As we know, in general relativity the notion of a finite three-dimensional volume of a medium generally loses its meaning.

One more remark is in order here. When processing the data of experiments in which energy imbalance has been observed, one must bear in mind that such imbalance may indicate the existence in the experiments of individual physical objects, bearers of the corresponding energies. The neutrino was discovered in this way.

We can say, therefore, that every physical model object has a characteristic, its energy, and this is always true irrespective of whether this characteristic is present in a theory or experiment explicitly or implicitly.

Now let us consider the second law of thermodynamics. For a system consisting of a large number of particles and described macroscopically, the concept of entropy plays a unique role, similar to that of energy in the first law. This macroscopic quantity, which reflects the peculiarities of thermal energy and irreversible processes, shows that statistically only such events are possible as lead to states with the highest probability. Both theory and experience show that states with the highest probabilities in many-particle small volumes of a given system are characterized by distributions with high peaks.

In constructing macroscopic models of physical objects described in a suitable way by the minimal number of characteristic quantities possible we may, and in general should, introduce, in accordance with the first and second laws of thermodynamics, both energy and entropy as postulated functions of a generally small number of parameters. It is clear that for many models these postulates can be substituted, on the basis of experiments or other preliminary theories, by equivalent postulates for other characteristics. Then we can express both energy and entropy in terms of these characteristics by means of calculations via formulas or in any other way.

In contrast to the notion of force in Newtonian mechanics, which is a vector with a restricted meaning and can be used in describing the interactions in only certain elementary mechanical systems and phenomena (i.e. interactions that do not include heat fluxes and fluxes of energy not related to the work of common forces, e.g. the magnetization and polarization of bodies, and many other interactions), the universal concept of energy for all types of objects is always linked to the idea of energy as a three-dimensional spatial scalar[2]. Interpreting the energy conservation law in a somewhat more complex manner, we can take the energy as a four-dimensional scalar in four-dimensional space with respect to position coordinates and time (this is essential in special relativity and general relativity).

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[2] A three-dimensional scalar is a quantity invariant under transformations of three spatial coordinates (which do not depend on time), while a four-dimensional scalar remains invariant under transformations of the general type of spatial coordinates and time.

The fact that energy is a scalar and the scalar nature of the equations of the first and second laws of thermodynamics are due to the covariance of the laws of physics, which is a natural generalization to global symmetry of the local symmetry laws represented by Galilean transformations in Newtonian mechanics and by Lorentz transformations in special relativity and general relativity.

Let us now recall the basic types of energy that appear in the form of arguments in thermodynamic functions and as separate terms in the first law of thermodynamics. This law is an outgrowth of Eq. (1.3), met with in studying physical phenomena in continuous media, matter and fields in which there may occur external actions and internal processes accompanied by transformations of energy from one type to another.

If we wish to fix the types of energy, their dependence on the main parameters, and the laws governing their transformation into each other, we must rely on the data concerning these types.

We will start with *nuclear energy*. This type manifests itself in strong and weak interactions of elementary particles. A vast body of data exists on this type of energy and also on the possible conditions in which it manifests itself. In many phenomena of physics and mechanics nuclear energy does not manifest itself in any way, and its presence is made evident only by the proper constants. However, there is no doubt that in many urgent problems of mechanics, and physics generally, that already face researchers the energy in atomic nuclei will be of primary importance.

The next type is *electromagnetic energy*. This is the main characteristic of electromagnetic fields. It manifests itself in structural microscopic collections of the atoms of molecules and larger associations, in various bodies and substances, in their interactions, and in chemical compounds of organic and inorganic materials. Electromagnetic energy plays an important role in problems associated with the propagation and absorption of various types of radiation, in emission of light waves and radio waves, in laser technology, etc. Obviously there are many mechanical effects directly related to and produced by electromagnetic interactions.

The third type, *chemical energy*, has an electromagnetic nature. It is associated with the energy conserved in fuels and explosives, the energy absorbed or liberated in chemical reactions in both inorganic and organic chemistry, and the like. It is clear that this type of energy is closely linked to the mechanical phenomena occurring in chemical processes.

Mechanical energy. Primarily this is the potential and the kinetic energy of relative motion related to the work of acting forces and moments of various order, the main ones being of gravitational and electromagnetic nature. The internal potential energy of continuous media also belongs to this type. In Newtonian mechanics we can consider the work and the respective energy associated with inertial forces.

The fifth type is *thermal energy*. This is the macroscopic energy of disordered structural states, microscopic processes, and internal motions in macroscopic bodies. We know that in many cases regular macroscopic types of energy, such as macroscopic mechanical energy, transform into thermal energy, which is highly important in irreversible phenomena. The properties of possible transformations of thermal energy into other types of energy are reflected by the second law of thermodynamics and serves as the basis for introducing such a macroscopic characteristic as entropy.

Finally, we turn to *gravitational energy*. In Newtonian mechanics gravitational energy is a four-dimensional scalar representing the energy of universal gravitation. This energy is ascribed to bodies and depends on their relative position. In general relativity the energy of a gravitational field is associated with the curvature of the Riemannian space, is determined by the geometric properties of four-dimensional space-time, and must be nonzero in free space, where there is no matter and fields but where there are gravitational waves propagating with a finite velocity.

Obtaining formulas that determine the energy in the simplest models is equivalent to finding the four-dimensional energy-momentum tensor $\mathcal{T}$. In this case the energy is simply the component $\mathcal{T}_4^{\;4}$, which is a three-dimensional scalar; in a coordinate transformation depending on the temporal coordinate the value of $\mathcal{T}_4^{\;4}$ may change.

To define energy still further we must introduce the notions of the following reference frames and the associated systems of coordinates.

1. The Lagrangian reference frame and coordinate system. This is a system of coordinates comoving with fixed individualized particles of the continuous medium (for matter and field) corresponding to which there are uniquely defined constant values of the spatial coordinates ξ^1, ξ^2, ξ^3 while the temporal coordinate ξ^4 changes along trajectories that do not intersect (the world line in four-dimensional space for each given individualized point in the medium). It is clear that different comoving systems of coordinates, ξ^i and η^i for a given reference frame are coupled by one-to-one transformations of the type

$$\eta^4 = f\,(\xi^1,\ \xi^2,\ \xi^3,\ \xi^4), \quad \eta^\alpha = \eta^\alpha\,(\xi^1,\ \xi^2,\ \xi^3), \quad \alpha = 1,\ 2,\ 3. \qquad (1.14)$$

2. The observer's reference frame and the respective system of coordinates x^i ($i = 1,\ 2,\ 3,\ 4$). The observer's reference frame must be given and may be fixed arbitrarily. In Newtonian mechanics the main type of observer's reference frame is an inertial frame. This is

the reference frame in which physical laws are usually stated. But as a rule, we know, we can also use an observer's reference frame that is not inertial and, in general, stressed. The observer's reference frame can always be thought of as a frame comoving with some abstractly introduced ideal media or objects.

In one and the same space the functional relationships of the type

$$x^i = f^i(\xi^1, \xi^2, \xi^3, \xi^4), \quad i = 1, 2, 3, 4, \qquad (1.15)$$

constitute the law of motion of the medium. In connection with the f^i, which are functions representing the law of motion, we can introduce various characteristics (velocity, acceleration, tensors representing strain and rate of strain, etc.), which may be used as arguments of the given and sought functions.

In the comoving reference frame of individualized points, the three-dimensional velocity and acceleration vectors at each point are always zero. In the observer's reference frame the three-dimensional velocities and accelerations of individual points are nonzero. On the other hand, in the comoving system of coordinates individual particles may become deformed with passage of time. It is clear that a comoving reference frame may be inertial only in certain cases.

However, for every point M in the medium and at each moment of time we can introduce a locally inertial observer with respect to which point M has a zero velocity. The corresponding inertial coordinate tetrad determined locally is called the proper system of coordinates. In the rest reference frame point M has a zero velocity but a nonzero acceleration, while the velocities of infinitely close points are generally nonzero but infinitely small.

The sets of inertial tetrads of rest reference frames for each point of the medium are determined at each moment of time, form a nonholonomic set of reference frames, and may be taken as an inherent characteristic of the noninertial comoving reference frame. We can consider the characteristic vectors and tensors by employing components and bases determined at each point in the rest reference frame.

The physical relationships between the characteristics of various phenomena established theoretically or experimentally in the local tetrads of the rest reference frames can be transformed to the observer's holonomic reference frame; this transformation constitutes solution of the basic problem of navigation [23, 24][3].

Thus, the fundamental equations of continuous medium mechanics (for one, thermodynamic equations) can be formulated for inertial proper reference frames in which at each given point M the medium is at rest. These equations and the various conclusions valid for the relative state of rest must then be transformed to the observer's reference frame by navigational calculations [23, 24].

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[3] The fundamental problem in navigational calculations in Newtonian mechanics was first formulated and solved for the general case by L.I. Tkachev in 1944 (see his doctor's thesis and book *Systems of Inertial Orientation* [in Russian], Moscow Power Institute, Moscow, 1973).

At the same time, the physical essence of a problem connected with the nonholonomic system of proper tetrads goes much deeper and, for one, is closely linked with the definition of energy as a four-dimensional scalar.

Indeed, consider two inertial laboratories A and B with identical measuring instruments but moving in relation to each other. We assume the earth to be the inertial reference frame A, while the other laboratory B is placed on the platform of a truck that is in translational motion along an ideal smooth road with a constant horizontal speed v. In laboratory B we take a small lead pellet that is at rest with respect to the platform. In this case the inertial reference frame connected with the platform will be the rest frame for the pellet and, obviously, in frame B the kinetic energy of the pellet will be zero in Newtonian mechanics while in special relativity the pellet's energy will be $E_0 = m_0c^2$, where m_0 is the rest mass of the pellet and c the speed of light.

On the other hand, in reference frame A connected with the earth the pellet's kinetic energy will be nonzero and equal to $m_0v^2/2$, while in special relativity the pellet's energy will again be $E_0 = m_0c^2$.

It can be shown that in special relativity the basic types of energy can always be introduced as four-dimensional invariants, while in Newtonian mechanics the kinetic energy is not a four-dimensional invariant. To rid ourselves of this situation with kinetic energy, in the energy balance equation for a small individualized particle we can take instead of the kinetic energy only the elemental work of external inertial forces equal to the increment of the kinetic energy, which because of the vector nature of Newton's second law of motion does not depend on the choice of the inertial system of coordinates.

The use of proper tetrads in system B is expedient because then the differential equations of the first and second laws of thermodynamics do not include the thermodynamically insignificant velocity $\mathbf{v}$ and kinetic energy (both of which depend on the choice of reference frame A); this as a rule does not depend on the processes studied in the proper reference frame or laws independent of the viewpoint of observer A, which can be chosen arbitrarily.

The above reasoning concerning the role of kinetic energy agrees with the traditional interpretation of the energy equation as a heat influx equation. The importance of the tetrad system B and of the corresponding form of the energy equations is manifested most vividly when in addition to mechanical energy we allow for other types of energy and develop expressions for these types.

The above frames of reference can be used to define mathematically a variety of physical concepts and characteristics expressed in terms of these concepts, characteristics with a kinematic, dynamical and general physical meaning. We note that each of the above-mentioned reference frames may help, on the one hand, to formulate and determine the corresponding four-dimensional tensor characteristics. On the other hand, characteristics of a scalar or tensor nature defined by certain conditions in a fixed reference frame can be considered in any other reference frame and, hence, may serve as characteristic quantities for phenomena and respective invariant formulations of various physical laws in any other systems of coordinates.

There are two reasons for determining physical characteristics (energy, for one) in tetrad reference frames (when this has been done, we can find these characteristics in any other reference frame):

(1) each tetrad at each point M in the medium is an inertial reference frame in which point M has zero velocity;

(2) for a given medium and each moment of time the system of tetrads defines a unique reference frame. To this we must add that experiments and their results are usually described in a local inertial reference frame.

What we have just said manifests itself with the greatest physical significance if we consider electromagnetic energy, which occurs as a separate term in the energy equation. To clarify this statement let us consider once more the laboratories in reference frames A and B while employing for each point in the medium the corresponding tetrads of the rest inertial platforms. Suppose that on platform B that moves with respect to A in an inertial reference frame there is an electrically charged sphere at the end of an ideal insulating rod (the sphere is made of conducting material and there are no objects that introduce perturbations). In this experiment measurement of the electromagnetic field by observer B both in Newtonian mechanics and in special relativity yields a stationary Coulomb field without a magnetic field, while measurement of the electromagnetic field by the "motionless" observer A yields a nonstationary electromagnetic field with a nonzero magnetic component. Thus, the results of freasurements by identical devices prove to be different in the two mames. The same is true for the energy density in free space outside the sphere; we know that the electromagnetic energy density in free space is

$$(E^2 + H^2)/8\pi. \tag{1.16}$$

Hence, the energy density in one and the same volume depends on the choice of the inertial reference frame, which in this experiment manifests itself most clearly. (For small three-dimensional velocities $\mathbf{v}$ of motion of frame B with respect to frame A the difference is very small and of the order v/c.) Here we must note that the electric and magnetic field vectors may be defined only in inertial reference frames, which in special relativity may be global, but in general relativity they are local near any point M, namely, the tetrad proper reference frames.

In this connection we recall that a comoving reference frame is not usually inertial, and the same is true of the observer's reference frame. As we know, in general relativity inertial tetrads can be introduced only locally, near each point in the medium.

On the other hand, **E** and **H**, which are characteristics of the electromagnetic field, can be defined only by measuring them in an inertia reference frame. Obviously, measuring devices that are identical but move in relation to each other yield different results[4]. In this connection there arises the problem of selecting an inertial reference frame that would guarantee unambiguous results in measurements of the physical characteristics of the electromagnetic field and definite expressions for the energy and its increments in the field. Here two main conditions must be met:

(a) We must fix the energy-momentum tensor of the electromagnetic field (we can also take the Minkowski tensor). The problem of fixing the electromagnetic field energy-momentum tensor was clarified in 1965 [8], although even until very recently there appeared works that confuse the picture. Now, however, a correct understanding of this problem seems to be gaining acceptance everywhere.

(b) To find the required energy fluxes at each point in the medium we must use the inertial reference frame, in which the medium is locally at rest (the rest frame).

These conditions ensure that the electromagnetic field energy is a four-dimensional scalar.

The foregoing analysis shows that the problem of establishing a correct picture of these phenomena and an expression for the electromagnetic energy, which enters into the first law of thermodynamics, is not trivial. All previous conclusions can be reduced to one condition, namely, that the energy density of the medium and electromagnetic field are represented by a four-dimensional scalar equal to the component T_4^4 of the energy-momentum tensor, the component being taken at each point in the medium in a rest inertial reference frame determined locally.

The situation with the energy of a gravitational field in general relativity is even more complex. Over a period of 70 years various ways were proposed for defining the energy of a gravitational field via the gravitational field energy-momentum pseudotensor. But pseudotensors defined in nonuniquely determined reference frames have no physical meaning. In one of his works the present author introduced a uniquely defined comoving system of coordinates for the Riemannian space. With this uniquely defined system he built a physically meaning real tensor (not a pseudotensor) of the energy-momentum of a gravitational field in general relativity [26], but the construction involves the possibility of various values for the invariant energy of the gravitational field. The quantity may be chosen, say, on the grounds of simplicity and then verified experimentally, but detecting the effects and gravitational waves experimentally poses as yet undetermined difficulties. This remark, true for any theory, is of a general nature, since in physics correspondence between well-constructed theories (which in themselves are important) and reality must always be checked and verified by experiments. In other words, every constructed model and all the numerical values of the model's parameters must correspond to experience.

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[4] Here there seems to be a rough analogy with the uncertainty principle of quantum mechanics, according to which the results of measurements depend not only on the state of the observer but also on the setting of the experiment and type of measuring devices. We know that in both special relativity and general relativity there are many quantities, e.g. geometric, whose measured values depend on the choice of a moving observer. These effects are similar to the Doppler effect, which is present in Newtonian mechanics.

These conclusions concerning the definition of energy and the methods that follow, which are linked with the universal thermodynamic laws, must be taken as the required theoretical basis for all future progress that uses previous experience and must clarify new generalizations in the light of achieved results.

In building models in general and continuous models in particular we must introduce the sought main varying characteristics and given constant parameters or given functions and also thermodynamic functions depending on them, such as the internal energy U, the free energy F, the entropy S, the laws for fluxes of various types of energy, and the like. In the process we must use the generalized equation reflecting the principle of virtual work or displacements, which are closely linked with those fundamental laws of nature, the first and second laws of thermodynamics; these laws for individualized small volumes dV_3 of particles and fields take the following form [5, 16]:

$$-\delta\,(U\,dV_3) + \delta A_m^{(e)} + \delta A_n^{(e)} + \delta Q^{(e)} + \delta Q^{**} = 0, \tag{1.17}$$

(the first law of thermodynamics),

$$-\rho\Theta\delta S\,dV_3 + \delta Q^{(e)} + \delta Q' = 0 \tag{1.18}$$

(the second law of thermodynamics), and

$$dm = \rho\,dV_3, \quad \delta\,dm = 0 \tag{1.19}$$

(the equation of continuity). Here $U\,dV_3$ is the total energy of the particles, $\delta\,(U\,dV_3)$ the virtual energy increment for imaginary phenomena, $\delta A_m^{(e)}$ and $\delta A_n^{(e)}$ the corresponding elementary virtual works for external body and surface forces (which in Newtonian mechanics must include the elementary work of inertial forces, $-dm\ (\mathbf{a}\cdot\delta\mathbf{r})$), $\delta Q^{(e)}$ the total virtual external influx of heat due to external distributed body sources and the external influxes of heat across the surface $d\Sigma$ encompassing the volume $dV_4 = dV_3\, dt$, δQ^{**} an additional virtual flux of nonthermal energy arising from the fact that the arguments of U and S contain various electromagnetic quantities or quantities expressible in terms of derivatives of the laws of motion and similar characteristics of matter and fields[5], and dQ' the uncompensated heat due to irreversible phenomena. Often in application the definition of dQ' uses ordinary linear relationships for the corresponding quantities in the Onsager theory or nonlinear relationships in the generalized Onsager theory, e.g. in the theory of plasticity or the theory of magnetic hysteresis.

Equation (1.19) is written on the assumption that nuclear processes play no role in real and virtual processes and hence

$$ddm = \delta\, dm = 0.$$

We also note that Eq. (1.18) is written under the assumption that there is an absolute temperature Θ.

By integrating Eqs. (1.17) and (1.18) over a four-dimensional volume V_4 we can replace them with integral relationships, which in general are not the integral laws for the energy of individualized objects as the initial equations are, since in integrating over V_4 the time intervals for different particles may be different, so that at different times different particles are inside V_4.

It immediately follows that the universal equations (1.17) and (1.18) can be used to derive the following basic variational equation [11, 13, 15]:

$$\delta \int\limits_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0, \tag{1.20}$$

where V_4 is an arbitrary four-dimensional finite volume of space-time. The Lagrangian $\Lambda = -U$ is the total thermodynamic energy density taken with a minus sign and generalized to all virtual processes that differ little from real processes[6]. This energy density is very close to the physical energy of matter and field in the varied volume V_4 built out of the volumes dV_4 of individual particles for appropriate moments of time.

We note here that Λ for virtual processes may contain additional terms that are zero for real processes. The variations δ are taken for constant Lagrangian coordinates ξ^1, ξ^2, ξ^3, ξ^4 and may be substituted by variations ∂ for fixed coordinates x^i in the observer's reference frame according to the operator formula [7]

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[5] The derivation of Eq. (1.17) and the justification of the term δQ^{**} were given by the present author in 1964 in a report [5] at the 11th International Congress on Applied Mechanics in Munich.

[6] To be more exact, Λ is the Lagrangian density, but for the sake of simplicity we will use the term Lagrangian.

$$\delta = \partial + \delta x^i \nabla_i. \tag{1.21}$$

The scalar equations (1.17)-(1.19) and Eq. (1.20), which follows from them, are written for arbitrary continuous variations of the sought quantities. This imposes certain restrictions on the coefficients of the variations—they must be zero after variation for all possible real processes and motions. This, in turn, leads to a closed system of Euler equations, equations of state, and additional conditions at strong discontinuities and the domain boundaries for the sought functions.

In the general theory the procedures by which such relationships are obtained and their analysis are well-known. From the covariance of the equations and laws of physics it follows that the Lagrangian Λ and the corresponding energy U must be introduced as four-dimensional scalars that generally depend on the known and, which is most important, on the sought functions, in particular, on functions that can be expressed in terms of the law of motion and certain physical parameters—the components of the sought tensor parameters μ^A (x^k) $(A = 1, 2, 3, \ldots)$ [10], the components of the metric tensor g_{ij}, and constant parameters or known functions K^B (ξ^k) or K^C (x^i) given in the comoving or observer reference frame, respectively. Since Λ is a scalar, its arguments can be written both in the comoving reference frame and in the observer's reference frame (or the tetrad system B).

If we use the basic variational equation (1.20), each of the three scalar terms has a definite physical meaning, where the volume V_4 is arbitrary and the variations of the sought functions on its boundary Σ may take arbitrary values.

In accordance with the fundamental equations of thermodynamics (1.17), (1.18), and (1.19), the variational equation has additional terms δW^* and δW for any volume V_4. In the general case the scalar functional δW^* is given by the formula

$$\delta W^* = \int\limits_{V_4} [\rho\Theta\, \delta S\, dV_3 - \delta Q' + \delta A_m^{(e)} + \delta Q_m^{**}]\,(dV_4/dV_3). \tag{1.22}$$

We see that δW^* depends on the influxes of heat $\delta Q^{(e)} = \rho\Theta\delta S\, dV_3 - \delta Q'$ and the body energy influxes $\delta A_m^{(e)} + \delta Q_m^{**}$, where δQ_m^{**} are the body influxes of energy due either to the variations of the additional parameters μ^A or K^B or to the presence of higher-order partial derivatives of the sought functions in the arguments of U.

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[7] The corresponding theory of variations that preserves the same tensor nature as the varied tensor component is not trivial. For more details see [31].

In analytic Newtonian mechanics one must introduce a term of the type δW^* when there are external nonpotential forces and nonholonomic constraints. In Newtonian mechanics the variation δW^* contains a term corresponding to the work of external inertial forces if Λ is expressed only in terms of the potential energy U.

In reversible phenomena $dQ' = 0$, while in irreversible dQ' is positive; for instance, in the presence of viscosity, electric currents, or hysteresis we can sometimes put [8]

$$\delta Q' = (\tau_i^{\ j} \nabla_j \delta x^i + I^k \partial A_k) dV_3, \tag{1.23}$$

where $\tau_i^{\ j}$ is the appropriate tensor of the type of viscous stress tensor, I^k the components of the four-dimensional current vector, and A_k the components of the four-dimensional vector potential of the electromagnetic field. The term $I^k \partial A_k$ is due to the fact that the phenomena associated with the interaction of the electromagnetic field and matter are nonconservative.

As to the scalar δW determined via Eq. (1.20), we find that since V_4 is arbitrary, for any variations on the boundary Σ the scalar δW is given by the formula

$$\delta W = \int_{\Sigma} (\mathscr{P}_i^{\ j} \delta x^i + M_A^j \delta \mu^A) n_j \, d\sigma, \tag{1.24}$$

where n_j are the components of a unit vector normal to Σ (this is true provided both Λ and δW^* have no derivatives of the sought functions with orders higher than the first). The components $\mathscr{P}_i^{\ j}$ and M_A^j are determined via Eq. (1.20). The tensor $\mathscr{P} = \mathscr{P}_i^{\ j} \mathbf{\partial}^i \mathbf{\partial}_j$ is simply the energy-momentum tensor of the field and medium. Both $\mathbf{\partial}^i$ and $\mathbf{\partial}_j$ are the coordinate vectors of a basis in the observer's reference frame, and the tensor components M_A^j define additional tensors similar to the tensor $\mathscr{P}$. The presence of M_A^j is due to energy fluxes through Σ, in view of the additional parameters μ^A in the arguments of Λ and in the expression for δW^*. The formulas for the components $\mathscr{P}_i^{\ j}$ and M_A^j are simply the thermodynamic equations of state.

We can also show that if inside V_4 there are surfaces $\tilde{\Sigma}$ of strong discontinuities, then on these surfaces there occur special conditions that reflect the continuity of the components $\mathscr{P}_i^{\ j} n_j (\tilde{\Sigma})$ and $M_A^j n_j (\tilde{\Sigma})$ on $\tilde{\Sigma}$ [16, 18] (if (1.24) holds).

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[8] The term $I^k \partial A_k$ in the formula for $\delta Q'$ first appeared in [29], and its justification was revealed in an improved version of this work (see [30], pp. 254-280).

Finally, what is most important is that the dynamical equations follow from (1.20) and are the Euler equations for the basic variational equation. We can easily show that the Euler equations for the variational equations (1.20) remain the same if for Λ we substitute $\Lambda + \nabla_i \Omega^i$, where Ω^i are any function of the determining parameters or other quantities [26]. But this substitution changes the expressions for the energy of the system and for the components $\mathcal{P}_i^{\,j}$ and M_A^j. In view of this, in any physical problem involving construction of a model the energy U and Lagrangian Λ must be given in full, notwithstanding the fact that terms of type $\nabla_i \Omega^i$ in the Lagrangian Λ do not influence the Euler equations. These remarks are important in solving the problem of the gravitational field energy-momentum tensor. This is obviously true in all cases where the building of models requires equations that must be used, say, in formulating the boundary conditions.

In the aforesaid we have established a procedure for building models that is based on very general physical principles and uses various information obtained through physical experiments and theories. For one, in the mechanics of continuous media the theories that are very fruitful are those of statistical physics. In these theories the thermodynamic functions are calculated in the rest reference frame using the simplest classical or quantum assumptions.

Of great importance is the various data obtained through processing the simplest experiments whose results can be given in the form of empirical formulas. These formulas are then generalized and are formulated in the test reference frame. We know of examples where empirical formulas become the basis of profound theoretical generalizations and new physical insights. We must also note the fruitfulness of Eq. (1.20) in developing various kinds of theoretical models, such as the theory of shallow water, the theory of various kinds of thin films, plates, shells, rods, and the like [21, 28, 32].

In all these examples we can reduce the dimensionality of the problem by finding a general plausible law of distribution of the sought functions of three variables in terms of functions of two or one variable with internal parametric functions of the reduced number of variables; for these we obtain the Euler equations, the appropriate equations of state, and boundary and other conditions. The Lagrangian of model problems with a reduced number of independent variables may be obtained, say, by integrating the Lagrangian related to a three-dimensional problem over a thin cross section (as is the case in the Pohlhausen method in the theory of a boundary layer). Such theories have been developed and the corresponding models have been substantiated on the basis of the basic variational equation (1.20).

With the help of Eq. (1.20) each of the known models and a number of new models can be obtained by using only the minimum of physically necessary and natural assumptions both in the framework of Newtonian mechanics and of special relativity and general relativity. The same is true even in a broader context of general ideas of space and time and, in particular, in microscopic theories of elementary particles [7, 9, 12, 14, 19, 20, 25, 29, 31]. However, the developed general theory based on Eq. (1.20) has not reached its full potential in the various applications.

Both in Newtonian mechanics and in special relativity and general relativity we may assume that the Lagrangian Λ is equal to $-U$; for instance, in Newtonian mechanics we may leave out the specific kinetic energy T. In this case, as we have shown above, δW^* must include the work of external inertial forces besides other external influxes of energy and the energy influxes of the type $\rho\Theta\delta S\, dV_4$ and $-\delta Q'$ due to irreversible phenomena.

In some cases we are able to write

$$\delta W^* = \delta \int\limits_{V_4} \Lambda^* dV_4 + \delta W^{*\prime}. \tag{1.25}$$

For this reason we may put

$$\Lambda = -U + \Lambda^* \tag{1.26}$$

instead of $\Lambda = -U$. But this changes the physical meaning of the Lagrangian as the energy U taken with a minus sign. In particular, according to Eq. (1.3), in analytic mechanics

$$\Lambda = T - U. \tag{1.27}$$

We could, following Eq. (1.11), put

$$\Lambda = -(T + U). \tag{1.28}$$

Such a reshuffling of terms in the basic variational equations in no way influences the Euler equations, but then the four-dimensional invariance of each of the terms in Eq. (1.20) is not retained although Eq. (1.20) as a whole remains a four-dimensional invariant.

If condition (1.27) is met, there is no explicit relation of the variational equation (1.20) to the first and second laws of thermodynamics. To this we may add that in special relativity and general relativity neither the kinetic energy in Λ nor the work of inertial forces in δW^* is present in Eq. (1.20). In contrast, in Newtonian mechanics there are many cases of adiabatic reversible processes where $dQ' = 0$ and $dS = 0$ and the condition (1.27) is met with $\Lambda^* = T$ and $\delta W^{*\prime} = 0$. But for irreversible processes in nonconservative systems the term $\delta W^{*\prime}$ must always be present in Eq. (1.20).

At present in physics, when building new models of media and fields (especially fields), the common form of the variational principle is

$$\delta \int\limits_{V_4^*} L\, dV_4 = 0, \tag{1.29}$$

where L is postulated from mathematical arguments and is not taken as the thermodynamic energy (at least explicitly); the volume V_4^* is taken fixed, and the variations of the sought functions are not tensor components and vanish on the boundary Σ^* of V_4^* (this is the volume of the region in which we are looking for a solution with given boundary conditions on Σ^*). The main problem is to build the Euler equations.

The possibility of obtaining the equations of state and the formulas for the energy-momentum tensor after we have fixed L and $\delta W^{*\prime} = 0$ or Λ and $\delta W^* \neq 0$ via Eq. (1.20) is often overlooked. In view of this, incorrect interpretations of the essence of the Euler equations and of the concept of the energy-momentum tensor are widespread.

Thus, we may say that in physics all constructions associated with the introduction of new models originate in one degree or another in integral variational equations but, unfortunately, without close links with the thermodynamics. At the same time we must note that in each separate case of building models one must overcome the difficulties associated with the complexity of describing the required assumptions and their interpretation.

Some simple models of the classical type for describing the motion of material media interacting with an electromagnetic field are discussed in [31]. In these models, Eq. (1.20) is used to derive, with thermodynamically justified Λ and δW^*, the Euler equations, which contain all the dynamical equations, including the Maxwell equations. For important cases we find formulas for the interaction between the electromagnetic field and matter, the forces being dependent on the field characteristics and the properties of matter, and other relationships.

The main meaning of the theoretical program outlined above consists in the fact that in both mechanics and physics there are only a few universal concepts and general relationships related to exact models specified by means of particular postulates. The concepts of energy and entropy, connected with the first and second laws of thermodynamics, may be considered basic to mechanics and physics. There are a few of such basic concepts, but understanding and applying them creatively require a thorough grasp of the essence of scientific methods of studying nature.

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2 Applying the Basic Variational Equation for Building Models of Matter and Fields

Academician *L. I. Sedov*

Institute of Mechanics, M. V. Lomonosov Moscow State University

INTRODUCTION

This paper offers a general variational method of building theoretical models for different substances and fields in an improved and more explicit form. The essence of the method is connected with a number of subtle points that must be brought out and taken into consideration in applications. Notwithstanding the large number of published works starting in the 1960s devoted to the basic variational equation, many details and a considerable part of the material presented here, material touching on the very basics of model construction, are published for the first time.

Many theories use variations of different quantities, sought or fixed, and their functions. Explicit definitions of mentally introduced variations may be quite different. The properties of and correlations between such variations for the same quantity, used in various holonomic or nonholonomic reference frames with indistinct designations, are often felt and used on the intuitive level in the mechanics of continuous media and physical fields and practically only in restricted theories (e.g. only for material media or special fields, when there are no geometric or physical variables that are parametrically fixed characteristic tensors, when the processes are reversible, etc., or when the observer uses Cartesian coordinates, which is impossible globally in Riemannian spaces).

Since the peculiar features of the variations used and the interrelationships between them are fundamental, there is a need to describe these peculiarities explicitly. This could simplify understanding the essence and sometimes even the peculiar nature of the usual theories.

2.1 DEFINITIONS

Suppose that in a metric space-time continuum of points, the coordinates ξ^i ($i = 1, 2, 3, 4$) with basis vectors $\hat{\text{Э}}_i$ and $\hat{\text{Э}}^i$ and with metric tensor components $\hat{g}_{ij}(\xi^k)$ form a comoving coordinate system, where the conditions $\xi^\alpha = \text{const}$, $\alpha = 1, 2, 3$, individualize the particles of the given ideal "medium"; the coordinates x^i with basis vectors $Э_i$ and $Э^i$ and with metric tensor components $g_{ij}(x^k)$ define the observer's system of coordinates, which can also be taken as a comoving system of coordinates for individual points with x^α = const for another ideal "medium".

Individualized world lines in the comoving system are coordinate lines along which the temporal coordinate ξ^4 changes, while in the observer's system the world lines correspond to the coordinate lines along which the temporal coordinate x^4 changes. Each family of world lines by definition forms a reference frame.

For the sake of simplicity we will assume that space-time is a four-dimensional Riemannian space (fixed or the one we are looking for) and has the local or global structure of the Minkowski space-time. However, this assumption is not always important.

A function that defines a global coordinate transformation in a four-dimensional space, say

$$x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4), \quad i = 1, 2, 3, 4, \tag{2.1.1}$$

is called a law of motion and may be considered for ξ^α = const, $\alpha = 1, 2, 3$, as an equation of the world lines of the comoving system in the x^i coordinates. Along with the law of motion (2.1.1) we may consider scalar, vector, or tensor fields defined in various ways and whose components can be defined via (2.1.1) or otherwise in the coordinate bases of the comoving system or the observer's system [1-5].

In problems that have to do with fixing or finding the laws of motion and fields we may use, aside from "real" laws of motion (2.1.1) and the above-mentioned "real" fields for the characteristic quantities, virtual laws of motion (introduced globally or locally, i.e. near each point) of a "medium" or fields in a fixed or varied four-dimensional space belonging to a certain class (e.g. the class of continuous Riemannian spaces) [6-10]. *Varied laws of motions and fields may be introduced in various ways depending on their definition* [6-13].

Variations of various functions may be considered as the increments of these functions, but the meaning of this operation must be clearly defined. For instance, the increment of a quantity defined inside a certain volume at each fixed point M may be considered as the result of moving from the given point M to a neighboring point M'. The variation of tensor components may be considered as their increments due to an infinitesimal transformation of the coordinate system at a fixed point M or otherwise. In contrast to common conceptions, there is no need to define variations as partial derivatives with respect to some auxiliary parameters that determine the global fields of the varied states and their characteristic functions.

The varied increments of given or sought quantities determined locally, i.e. near each point inside a restricted region, can be defined via nonholonomic relations. The physical meaning of given functionals, e.g. δW^* (a functional in the basic equation; see the text below), in real phenomena as well as in varied states must determine the mathematical form of the variations of the characteristic parameters in terms of which one expresses the functionals.

In what follows we will consider none but infinitely small variations of scalars and tensors. Some important particular cases of correlated variations in a fixed system of coordinates are considered together with the case of arbitrary variations, coupled, however, in the coordinate systems ξ^i and x^i.

We also note that in an infinitesimal variation of a tensor it is expedient (so as to retain the properties of invariance in all relations that we obtain) to use infinitesimal tensors of the same mathematical nature as the variations. (Many authors using the variation technique do not devote enough attention to this point.)

Thus, together with functions of type (2.1.1) we introduce varied functions

$$x'^i = x^i(\xi^k) + \delta x^i(\xi^k), \tag{2.1.2}$$

with δx^i the infinitesimal virtual displacements of points M of the world lines into neighboring points M' in space-time (these displacements are measured in the observer's system of coordinates). The variations $\delta x'$ are components of a vector field $\delta \mathbf{r} = \delta x^i Э_i$, which field can be considered both in the observer's system and in the comoving system:

$$\delta \mathbf{r} = \delta x^i Э_i = \delta \hat{x}^i \hat{Э}_i, \quad \text{where} \quad \delta \hat{x}^i = \delta x^k \, \partial \xi^i / \partial x^k. \tag{2.1.3}$$

In addition we may assume by definition that $\delta_d \mathbf{r}$ for real processes becomes a real displacement, i.e.

$$\delta_d \mathbf{r} = d\mathbf{r} = \mathbf{u}\, ds, \qquad d\hat{x}^4 = ds/\sqrt{\hat{g}_{44}}, \qquad d\hat{x}^\alpha = 0,$$

where $\mathbf{u}$ is the velocity four-vector. i.e. a unit vector tangent to a world line in the comoving reference frame, and ds is an element of these world lines.

For varied basis vectors $Э_i$ and $Э^i$ in any system of coordinates we may write for each point M

$$Э'_i = Э_i + \omega_i{}^k Э_k, \quad Э'^j = Э^j + \Omega^j{}_s Э^s, \tag{2.1.4}$$

where $\omega = \omega_i{}^k Э^i Э_k$ and $\Omega = \Omega^j{}_s Э_j Э^s$ are infinitesimal second-rank tensors. To retain the scalar product for two vectors, one with covariant components and the other with contravariant, and the main property of covariant and contravariant basis vectors in the varied form, we must assume that

$$Э'_i \cdot Э'^j = \delta_i^j, \tag{2.1.5}$$

where δ_i^j is the Kronecker delta. This implies that

$$\omega_i{}^k = -\Omega^k{}_i,$$

whence in the general case formula (2.1.4) takes the form

$$\text{Э}'_i = (\delta_i^k - \Omega^k{}_i)\,\text{Э}_k, \quad \text{Э}'^j = (\delta_k^j + \Omega^j{}_k)\,\text{Э}^k. \tag{2.1.6}$$

Many formulas below have nothing to do with raising or lowering the indices of the $\Omega^p{}_q$ components, but we may assume that juggling of the indices of the components of tensor Ω is achieved via the covariant and contravariant components of the metric tensor, $g_{ij} = (\text{Э}_i \cdot \text{Э}_j)$ and $g^{ij} = (\text{Э}^i \cdot \text{Э}^j)$.

2.2 VARIATIONS OF TENSORS FOR WHICH SCALAR INVARIANTS RETAIN THEIR FORM

Let us take some examples in which the variations of covariant or contravariant tensor components are nonzero while the variations of the scalar products are zero.

Suppose that we have two vectors $\mathbf{A} = A^i\text{Э}_i$ and $\mathbf{B} = B_i\text{Э}^i$ with the scalar product $\mathbf{A}\cdot\mathbf{B} = A^iB_i$. By varying the components A^i and B_i and the basis vectors according to (2.1.6) we get

$$\mathbf{A}' = A'^i\text{Э}'_i = A'^k\,(\delta_k^i - \Omega^i{}_k)\,\text{Э}_i,$$

$$\mathbf{B}' = B'_i\text{Э}'^i = B'_k\,(\delta_i^k + \Omega^k{}_i)\,\text{Э}^i.$$

If we also put

$$A'^i - A^i = \delta A^i \quad \text{and} \quad B'_i - B_i = \delta B_i,$$

where $\delta A^i\text{Э}_i$ and $\delta B_i\text{Э}^i$ are infinitesimal vectors[1], we can write

$$\begin{aligned} \delta\mathbf{A} &= A'^i\text{Э}'_i - A^i\text{Э}_i = \delta A^i\text{Э}_i - A^k\Omega^i{}_k\text{Э}_i = \delta^* A^i\text{Э}_i, \\ \delta\mathbf{B} &= B'_j\text{Э}'^j - B_j\text{Э}^j = \delta B_j\text{Э}^j + B_k\Omega^k{}_j\text{Э}^j = \delta^* B_j\text{Э}^j. \end{aligned} \tag{2.2.1}$$

Here the tensor components $\Omega^p{}_q$ are arbitrary.[2]

The above definitions imply that

$$A'^i\text{Э}'_i - A^i\text{Э}_i \neq A'_j\text{Э}'^j - A_j\text{Э}^j.$$

The form of (2.2.1) is retained both when the basis vectors of the coordinate system are varied ($\Omega^k{}_i \neq 0$) and when the components A^i and B_j are varied in a fixed system of coordinates.

---

[1] Both δA^i and δB_i are vector components if the $\Omega^p{}_q$ are tensor components, which follows from (2.2.1).

[2] In (2.2.1) the terms depending on the $\Omega^p{}_q$ may emerge, for one, from δA^i and δB_j, irrespective of whether the basis vectors are varied or not.

Formulas (2.2.1) imply that

$$A'^i B'_i = A^i B_i + \delta\,(A^i B_i),$$

which does not contain the $\Omega^i{}_k$ components. Thus, the variation of the scalar product of two vectors is zero if $\delta A^i = \delta B_i = 0$ but $\delta^* A^i$ and $\delta^* B_i$ are nonzero.

For the variation of tensors of arbitrary rank, e.g. tensor $\mathbf{T} = T^i{}_j{}^k Э_i Э^j Э_k$, formulas (2.2.1) transform to

$$\begin{aligned}\delta^* T &= (\delta T^i{}_j{}^k)\, Э_i Э^j Э_k - T^p{}_j{}^k \Omega^i{}_p Э_i Э^j Э_k \\ &\quad + T^i{}_p{}^k \Omega^p{}_j Э_i Э^j Э_k - T^i{}_j{}^p \Omega^k{}_p Э_i Э^j Э^k \\ &= (\delta T^i{}_j{}^k + \delta_\Omega T^i{}_j{}^k)\, Э_i Э^j Э_k. \end{aligned} \qquad (2.2.2)$$

Obviously, for any scalar contraction of tensors, e.g. $T^i{}_j{}^k T_i{}^j B_k$, for all $\Omega^p{}_q$ we have

$$\delta^*\,(T^i{}_j{}^k T_i{}^j B_k) = \delta\,(T^i{}_j{}^k T_i{}^j B_k). \qquad (2.2.2')$$

Since all scalar invariants of a system of tensors are expressible as functions of an appropriate system of scalar contractions formed from the components of these tensors, we conclude that the variation components δ_Ω of tensor components can be expressed in terms of mixed tensor components $\Omega^p{}_q$ related to variations of the basis vectors $Э_i$ and do not influence the variations of any scalar.

Scalar contractions besides the sought variable components of tensors may contain given, variable or fixed, components of parametric tensors. The components of such parametric tensors must be varied as long as we consider the variations δ_Ω of the tensor components due to transformations obtained as a result of varying the $Э_i$ and $Э^i$.

Thus, if we vary a scalar L depending on tensor components for which the infinitesimal variations δ_Ω are expressed only in terms of $\Omega^p{}_q \neq 0$ via (2.2.2), we find that the variations of L vanish for any infinitesimal $\Omega^p{}_q$ irrespective of the type of function L. If the varied quantity L depending on tensor components is not a scalar but is invariant under transformations of a group G, the above statement that $\delta_\Omega L = 0$ remains valid if the tensor components $\Omega^p{}_q$ define the variation of bases corresponding to transformations of G. For the components of the metric tensor we may write

$$\delta^* g_{ij} = \delta g_{ij} + \Omega^p{}_i g_{pj} + \Omega^p{}_j g_{pi} = \delta g_{ij} + \Omega_{ji} + \Omega_{ij} \qquad (2.2.3)$$

and similarly

$$\delta^* g^{ij} = \delta g^{ij} - \Omega^{ij} - \Omega^{ji}\,. \qquad (2.2.4)$$

This implies that $\delta^* g_{ij} = \delta g_{ij}$ and $\delta^* g^{ij} = \delta g^{ij}$ if the Ω_{ij} are antisymmetric, i.e. $\Omega_{ij} = -\Omega_{ji}$; but if the Ω_{ji} are arbitrary,

$$\delta_\Omega g_{ij} = -g_{pi} g_{qj} \delta_\Omega g^{pq}. \qquad (2.2.5)$$

2.3 SPECIAL TYPES OF TENSOR COMPONENTS $\Omega^i{}_j$

Here are some important formulas for the $\Omega^p{}_q$ that follow from natural assumptions.

Example 1. In problems involving the law of motion (2.1.1) we can write

$$\hat{Э}_j = (\partial x^i/\partial \xi^j)\, Э_i \quad \text{and} \quad \hat{Э}^i = (\partial \xi^i/\partial x^j)\, Э^j. \tag{2.3.1}$$

We introduce the notations

$$\partial x^i/\partial \xi^j = x^i{}_j, \quad \partial \xi^i/\partial x^j = \xi_j{}^i. \tag{2.3.2}$$

For a fixed index j and, correspondingly, for the ξ^i constant, the $x^i{}_j$ may be taken as the components of vector $\hat{Э}_j$ in the basis $Э_i$.

According to (2.3.1) we can write

$$\delta\hat{Э}_j = \delta\,[(\partial x^i/\partial \xi^j)\, Э_i] = (\delta'\, \partial x^i/\partial \xi^j)\, Э_i + x^i{}_j\, (\delta' Э_i + \delta_\Omega Э_i). \tag{2.3.3}$$

The variation $\delta\hat{Э}_j$ is taken at the ξ^i constant, and $\delta'\,(\partial x^i/\partial \xi^j)$ is a variation with respect to $x^i{}_j\,(\xi^k)$ at the ξ^k and $Э_i$ constant. Whence

$$\delta' x^i{}_j = \partial \delta x^i\,(\xi^k)/\partial \xi^j = x^s{}_j\, \partial \delta x^i\,(x^k)/\partial x^s. \tag{2.3.4}$$

Next, $\delta' Э_i$ is the variation of vector $Э_i$ due to the displacement of point $M(x^k)$ to point $M\,(x^k + \delta x^k)$ for a fixed point with coordinates ξ^i, while $\delta_\Omega Э_i$ is the variation of $Э_i$ at a fixed point with coordinates ξ^i and x^i due to a change in the basis vectors $Э_i$. Consequently,

$$\delta' Э_s = \Gamma^i_{sk}\,\delta x^k Э_i \quad \text{and} \quad \delta_\Omega Э_s = -\Omega^i{}_s Э_i. \tag{2.3.5}$$

On the basis of (2.3.4) and (2.3.5) we can write the tensor relationship (2.3.3) in the following form:

$$-\hat{\Omega}^k{}_j \hat{Э}_k = \delta\hat{Э}_j = x^s{}_j\, [\nabla_s \delta x^i - \Omega^i{}_s]\, Э_i. \tag{2.3.6}$$

When $\nabla_s \delta x^i \neq 0$, this implies that $\delta\hat{Э}_j \neq 0$ at $\Omega^i{}_s = 0$ and that

$$\delta x^i{}_j = x^s{}_j \nabla_s \delta x^i; \tag{2.3.7}$$

if $\delta\hat{Э}_j = 0$ (and, respectively, $\delta\hat{Э}^i = 0$; see below), we have

$$\Omega^i{}_s = \nabla_s \delta x^i. \tag{2.3.8}$$

On the basis of (2.3.8) we can rewrite (2.2.3) and (2.2.4) in the following form:

$$\delta\,(g_{ij}Э^iЭ^j) = (\delta g_{ij})\,Э^iЭ^j + 2\varepsilon_{ij}Э^iЭ^j,$$
$$\delta\,(g^{ij}Э_iЭ_j) = (\delta g^{ij})\,Э_iЭ_j - 2\varepsilon^{ij}Э_iЭ_j,$$

where the ten independent components of the symmetric tensor of infinitesimal stresses ε_{ij} can be expressed in terms of the four components δx^i of the displacement vector. Obviously, the components ε_{ij} at a given point can take on arbitrary values, but being functions of coordinates they must satisfy the well-known compatibility conditions.

Example 2. Now let us take a fixed point M and at this point consider an infinitesimal transformation of coordinates:

$$x'^i = x^i + \delta\eta^i\,(x^k). \tag{2.3.9}$$

Here x^i and x'^i are the initial and transformed coordinates at the same point M. We introduce the variations

$$g'_{pq}Э'^pЭ'^q - g_{ij}Э^iЭ^j = \tilde{\delta} g_{ij}Э^iЭ^j .$$

If we employ the well-known formulas for transforming tensor components and basis vectors under a transformation of coordinates,

$$g'_{pq} = g_{ij}\frac{\partial x^i}{\partial x'^p}\frac{\partial x^j}{\partial x'^q}, \quad Э'^pЭ'^q = \frac{\partial x'^p}{\partial x^i}\frac{\partial x'^q}{\partial x^j}\,Э^iЭ^j. \tag{2.3.10}$$

then after we have substituted (2.3.10) into the definition of $\tilde{\delta} g_{ij}$ we arrive at the trivial result

$$\tilde{\delta} g_{ij} = 0.$$

To have a nonzero $\tilde{\delta} g_{ij}$ and yet retain the tensor nature of $\tilde{\delta} g_{ij}$, we change the second formula in (2.3.10) for $Э'^k$ in the following way:

$$Э'^k = Э^k + (\partial Э^k/\partial x^r)\,\delta\eta^r = Э^k - \Gamma^k_{rl}\,\delta\eta^r Э^l$$

and, respectively,

$$Э'_k = Э_k + \Gamma^l_{rk}\,\delta\eta^r Э_l. \tag{2.3.11}$$

Now, bearing in mind that

$$\partial x^m/\partial x'^r = \delta^m_r - \partial\delta\eta^m/\partial x^r, \quad \partial x'^m/\partial x^r = \delta^m_r + \partial\delta\eta^m/\partial x^r, \tag{2.3.12}$$

and the formulas (2.3.10) for g'_{pq} and (2.3.11) for $Э'^k$, we find that to within infinitesimals of the first order

$$\tilde{\delta} g_{ij} = -\nabla_i\delta\eta_j - \nabla_j\delta\eta_i,$$

and in a similar way we find that

$$\delta g^{ij} = \nabla^i \delta\eta^j + \nabla^j \delta\eta^i. \tag{2.3.13}$$

Obviously, in this case

$$\tilde{\delta} g_{ij} = -g_{ip} g_{jq} \tilde{\delta} g^{pq}, \tag{2.3.14}$$

which can immediately be generalized to arbitrary tensors of any rank and with an arbitrary index structure.

Formula (2.3.13) and its generalization to arbitrary tensors imply that the transformation (2.3.9) under assumption (2.3.11) corresponds to an $\Omega^j{}_i$ represented by the formula

$$\Omega^j{}_i = -\nabla_i \delta\eta^j. \tag{2.3.15}$$

The formulas (2.3.8) and (2.3.15) are different notwithstanding the fact that the δx^i enter formulas (2.1.2) and the $\delta\eta^i$ similar formulas (2.3.9) in the same way. In both cases formula (2.2.2) retains its form but the $\Omega^j{}_i$ prove to be different.[3] Formulas (2.3.15) impose differential restrictions on the variations expressible in terms of $\delta\eta^j$.

To find the corollaries of the fact that a function is a scalar, we must use the variations of the arguments of any parameters of a tensor nature defined by the general formula (2.3.5) for the same arbitrary components $\Omega^j{}_i$ or for $\Omega^j{}_i$ defined by (2.3.15). After we have varied the scalar with respect to all the tensor arguments on which it depends and nullified in the sum obtained all coefficients of the arbitrary $\Omega^j{}_i$, e.g. those defined by (2.3.15), we arrive at 16 relationships that are valid identically. The number and type of identities can be changed as we integrate over the volume, with separating terms that vanish if the Gauss theorem (the divergence theorem) is applied.

We note once more that in obtaining the above identities we must vary all tensor arguments (both constant and variable) whose values depend on the choice of the coordinate system.

2.4 DEFINING VARIATIONS AND THEIR INTERRELATIONSHIP IN THE COMOVING AND THE OBSERVER'S REFERENCE FRAME

Obviously, if we wish to obtain equations by varying a scalar L and not simply identities that follow from the fact that L is a scalar, we can exclude the variations of the basis vectors and tensor components that are due to the tensor $\Omega^j{}_i$. On the other hand, if we wish to obtain such identities, it suffices to vary $\delta_\Omega L$ in such a way that the variations of all tensor arguments are expressed solely in terms of the $\Omega^j{}_i$ via the transformations of the coordinate bases according to (2.2.2′).

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[3] The above analysis enables us to explicitly realize the essence of the various ways in which variations connected with displacements and transformations of coordinates are defined.

Let us now consider variations of the general type, where in a given coordinate system no variations of the type δ_Ω exist. First we will give the definition of a variation of a scalar L. Thus, we start with a scalar L that can be studied and varied either directly or through its arguments, which are either unknown or fixed functions of coordinates x^i in the observer's reference frame, or by using the law of motion (2.1.1) and expressing the arguments as functions of coordinates ξ^k in the comoving system. According to

$$L\,(x^i\,(\xi^k)) = L\,(\xi^k);\quad L'\,(x^i\,(\xi^k) + \delta x^i\,(\xi^k)) = L'\,(\xi^k)$$

we can write

$$\begin{aligned}\delta L &= L'\,(\xi^k) - L\,(\xi^k)\\ &= L'\,(x'^i) - L\,(x'^i) + L\,(x^i + \delta x^i) - L\,(x^i),\end{aligned}$$

or

$$\delta L = \partial L + \delta x^i \nabla_i L \quad (\partial L/\partial x^i = \nabla_i L), \tag{2.4.1}$$

since

$$L'\,(x'^i) - L\,(x'^i) \approx L'\,(x^i) - L\,(x^i)$$

to within higher-order infinitesimals. Here the symbol δ corresponds to variations with constant ξ^k and the symbol ∂ to variations with constant x^k. Obviously, for real phenomena the variations ∂ with respect to any quantities are zero by definition.

Formula (2.4.1) gives the interrelationship between arbitrary variations of scalars in the comoving system and the observer's system.

When we go from a mentally varied phenomenon to the realized phenomenon, the real process, we may use the two definitions of variations given below.

1. Mentally introduced variations, e.g. δL, are virtual, which means that, by definition, for fixed ξ^i these variations vanish when we go over to real processes or states. Hence, for real processes we have

$$x'^i = x^i \quad \text{and} \quad \delta x^i = 0$$

and whence

$$\delta L = \partial L = 0. \tag{2.4.2}$$

2. Mentally introduced variations at a given point ξ^i, which we denote simply by δ_d, are possible and are given by the relationships

$$\delta_d x^i = dx^i + \delta x^i, \quad \delta_d L = dL + \delta L; \tag{2.4.3}$$

etc. Here dx^i, dL, etc. are infinitesimal increments of x^i, L, etc. in real processes. Whence, when we go over to real processes, we have

$$\delta_d x^i = dx^i, \quad \delta_d L = dL = (\partial L/\partial\tau)\, d\tau, \tag{2.4.4}$$

where τ is proper time. Hence, in such a definition of variations, when we go over from varied phenomena to real, the variations δ_d become the real increments of the varied quantities, which generally differ from zero.

If some quantities $\mu^1, \mu^2, \ldots, \mu^n$ are coupled by equations, the variations $\delta\mu^k$ satisfy these equations with fixed coordinates ξ^i that enter the equations as parameters, while the possible variations $\delta_d\mu^k$ satisfy relations which allow for a nonzero increment in the temporal coordinate. For instance, for a holonomic constraint $f(\mu^1, \mu^2, \ldots, \tau) = 0$ and at $\delta\tau = 0$ we have

$$(\partial f/\partial\mu^1)\, \delta\mu^1 + (\partial f/\partial\mu^2)\, \delta\mu^2 + \ldots + (\partial f/\partial\mu^n)\, \delta\mu^n = 0$$

and

$$(\partial f/\partial\mu^1)\, \delta_d\mu^1 + (\partial f/\partial\mu^2)\, \delta_d\mu^2 + \ldots + (\partial f/\partial\tau)\, d\tau = 0,$$

while for a nonholonomic constraint we have

$$A_1(\mu^1, \mu^2, \ldots, \xi^i)\, \delta\mu^1 + A_2(\mu^1, \mu^2, \ldots, \xi^i)\, \delta\mu^2 + \ldots + A_n\delta\mu^n = 0$$

and

$$A_1\delta_d\mu^1 + A_2\delta_d\mu^2 + \ldots + A\, d\tau = 0.$$

From the fact that $dL = dx^i\nabla_i L$ it follows that (2.4.1) for the same ∂L is valid for both virtual variations δ and possible variations δ_d.

The formula (2.4.1) considered above for the variations of scalar L are naturally generalized to variations of tensor components in the comoving coordinate system with fixed bases $\hat{\text{Э}}_i$ and $\hat{\text{Э}}^i$. The result can then be transformed into the observer's reference frame. We can then use the variations of tensor components in the observer's reference frame, defined in a similar way for the components of the same tensor, the components being transformed from bases $\hat{\text{Э}}_i$ and $\hat{\text{Э}}^i$ to bases Э_i and Э^i.

For simplicity we will first define the variations of the components of vector

$$\boldsymbol{\mu} = \hat{\mu}^i(\xi^k)\, \hat{\text{Э}}_i = \hat{\mu}_j(\xi^k)\, \hat{\text{Э}}^j = \mu^i(x^k)\, \text{Э}_i = \mu_j(x^k)\, \text{Э}^j,$$

where

$$\hat{\mu}_j = \hat{g}_{ji}\hat{\mu}^i, \quad \mu_j = g_{ji}\mu^i, \quad \mu^i = x^i{}_j\hat{\mu}^j, \tag{2.4.5}$$

and, respectively, the variations of the components of tensors of arbitrary rank and with arbitrary index structure. According to (2.4.5), for given or sought $\hat{g}_{ij}$, g_{ij}, and $x^i{}_j$ (ξ^k), which are defined separately either in the statement of the problem or in its solution, it is sufficient to fix or define one of the functions $\hat{\mu}^i$ or μ^i or $\hat{\mu}_i$ or μ_i so as to find all the other. In this connection, it is clear that for any sought tensor it is sufficient and generally necessary to formulate the problem for the tensor components with only one definite index structure. The components with a different index structure can then be found via (2.4.5). If we use for arguments of each tensor only the components with a single index structure in a fixed basis, there is no need for the formulas in (2.4.5). These may be needed only post factum to define the respective components with other index structure.

In varying tensors it is expedient to consider their varied components as tensors of the same nature but in unvaried bases. For this reason the varied tensor components will obey relations of the type (2.4.5), where the $\hat{g}_{ij}$, g_{ji}, and $x^i{}_j$ are taken for real phenomena. The formulas *cannot be used* for variation. It is clear that formulas (2.4.5) are also valid for increments of tensor components defined for real solutions.

In determining variations we find that the components of $\delta\boldsymbol{\mu}$ and $\partial\boldsymbol{\mu}$ in various bases and with various index structure are also related by formulas of the type (2.4.5). In other words, when constructing varied functions, we must not vary $\hat{g}_{ij}$, g_{ji}, and $x^i{}_j$ in (2.4.5). This implies that the varied values of tensor components with any index structure and in various bases are determined in a one-to-one manner by components with a fixed index structure.

For instance, in the comoving reference frame we have, by definition,

$$\delta\boldsymbol{\mu} = \delta\hat{\mu}^i\hat{\mathbf{Э}}_i = \delta\hat{\mu}_j\mathbf{Э}^j,$$

with $\delta\hat{\mu}_j = \hat{g}_{ji}\delta\hat{\mu}^i$ and $\delta\hat{\mu}^i = \hat{\mu}'^i(\xi^k) - \hat{\mu}^i(\xi^k)$. Next, we determine $\mu'^i(x^k)$ from the obvious necessary condition

$$\begin{aligned}\delta\boldsymbol{\mu} &= [\partial\mu^i\ (x^k) + \delta x^p\nabla_p\mu^i\ (x^k)]\ \mathbf{Э}_i\\ &= [\partial\mu_j\ (x^k) + \delta x^p\nabla_p\mu_j\ (x^k)]\ \mathbf{Э}^j,\end{aligned} \tag{2.4.6}$$

with

$$\mu_j = g_{ji}\mu^i\ (x^k) \quad \text{and} \quad \partial\mu_j = g_{ij}\,\partial\mu^i.$$

The presence of the covariant derivatives ∇_p with respect to x^p in (2.4.6) is an important fact since in the arguments of μ'^i and μ'_j there is a shift from point M to point M' owing to the variations δx^i.

From (2.4.5) and (2.4.6) in the observer's reference frame we have

$$\delta\mu^i\,(\xi^k) = \partial\mu^i\,(x^k) + \delta x^p \nabla_p \mu^i\,(x^k) \equiv x^i{}_s \delta\hat{\mu}^s\,(\xi^k), \tag{2.4.7}$$

and similarly

$$\delta\mu_j\,(\xi^k) = \partial\mu_j\,(x^k) + \delta x^p \nabla_p \mu_j\,(x^k). \tag{2.4.8}$$

We must note at this point that the definition of the variations $\delta\mu$ is given in the comoving system of coordinates ξ^i, while the definition of the variations $\partial\mu$ is given in the observer's system of coordinates x^i (which may be arbitrary). However, the vector $\partial\mu$, whose components may be taken in any system of coordinates including the comoving, depends essentially on the choice of the observer's system of coordinates, since its nature is similar to that of the observer's basis vectors. These are real vectors and may be taken in any reference frame, but essentially they are linked to the choice of the main reference frame of the observer. Obviously, (2.4.6)-(2.4.8) just as (2.4.1) are valid and retain their form both for δ and for δ_d.

We can see that for variations of the tensor components $T^i{}_j{}^k$ the generalization of formulas (2.4.7) and (2.4.8) has the form

$$\delta T^i{}_j{}^k = \partial T^i{}_j{}^k + \delta x^p \nabla_p T^i{}_j{}^k\,(x^k). \tag{2.4.9}$$

In the same way as in varying scalars, when we go over from virtual processes and, respectively, from possible processes to real processes, we can use the variations defined above in item 1 or 2.

On the other hand, such quantities as $x^i{}_j$, $\hat{g}_{ij}$, and g_{ij} may occur explicitly as sought or given characteristics in the form of separate arguments, similar to the varied components μ^i discussed above; hence, they must also be varied in the sense discussed above.[4] In such cases we must use formulas (2.4.6)-(2.4.8) if we wish to operate with variations of such additional arguments. The following formulas, besides (2.3.7), become useful in operating with the $x^i{}_j$, which occur as separate arguments.

Let us take the second-rank tensor g, which can be written in the following form if we account for transformations of basis vectors:

$$g = \frac{\partial x^i}{\partial \xi^j}\,\text{Э}_i \hat{\text{Э}}^j = \delta^k_j \hat{\text{Э}}_k \hat{\text{Э}}^j = \delta^i_k \text{Э}_i \text{Э}^k.$$

This means that the second-rank tensor **g** is the same in all reference frames.[5]

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[4] If the sought functions g_{ij} and g^{ij} also belong to the argument of the varied functions, we must substitute g^{ij} according to the following formulas: $g^{ij} = (1/g)\,(\partial g/\partial g_{ij})$, where $g = \det \| g_{ij} \|$. The contravariant components of the tensor ∂g_{ij} must be calculated via formulas of the type (2.4.5).

[5] The reader can easily see that in view of (2.3.7′)

$$\delta_g = \delta_\Omega\,[x^i{}_j \text{Э}_i \hat{\text{Э}}^j] = 0.$$

On the basis of (2.3.7) and (2.4.7), where the variations $\partial x^i{}_j$ are assumed to be vector components, we obtain

$$\delta x^i{}_j = x^s{}_j \nabla_s \delta x^i = \partial x^i{}_j + \delta x^s \nabla_s x^i{}_j. \tag{2.4.10}$$

From this follows an important formula for tensors:

$$\partial x^i{}_j = x^s{}_j \nabla_s \delta x^i - \delta x^s \nabla_s x^i{}_j. \tag{2.4.11}$$

The reader can easily see that (2.4.11) is equivalent to

$$\partial x^i{}_j = x^s{}_j\, \partial \delta x^i / \partial x^s - \delta x^s\, \partial x^i{}_j / \partial x^s, \tag{2.4.12}$$

where

$$x^s{}_j\, \partial \delta x^i / \partial x^s = \partial \delta x^i / \partial \xi^j = \delta' x^i{}_j \quad \text{and} \quad \delta x^s\, \partial x^i{}_j / \partial x^s$$

are not tensors. We note that $\delta' x^i{}_j \neq \delta x^i{}_j$, in accordance with the definitions on page 48. For real processes $\partial x^i{}_j = 0$.

2.5 AUXILIARY FORMULAS FOR VARIATIONS

Besides the formulas in Sec. 2.4 we list the following useful formulas:

$$\delta g_{ij} = \delta_d g_{ij} = \partial g_{ij}, \quad dg_{ij} = dx^s \nabla_s g_{ij} = 0.$$

We also have

$$\partial\, (\partial g_{ij} / \partial x^k) = \partial\, (\partial g_{ij}) / \partial x^k,$$
$$\partial\, (\partial^2 g_{ij} / \partial x^k \partial x^l) = \partial^2\, (\partial g_{ij}) / \partial x^k \partial x^l.$$

This implies for variations of Christoffel symbols the following:

$$\begin{aligned}\partial \Gamma^k_{ij} &= (1/2)\, g^{ks} (\nabla_i\, \partial g_{sj} + \nabla_j\, \partial g_{si} - \nabla_s\, \partial g_{ij}) \\ &= (1/2)\, [g^{kn} (\delta^s_i \delta^q_j + \delta^s_j \delta^q_i) - g^{ks} \delta^n_i \delta^q_j]\, \nabla_s\, \partial g_{nq}\end{aligned}$$

and

$$\partial \nabla_i \mu^A = \nabla_i\, \partial \mu^A + F^{Aj}_{Bs} \mu^B\, \partial \Gamma^s_{ij}. \tag{2.5.1}$$

Here A stands for all the indices in components of a tensor of arbitrary rank μ^A, and the corresponding factor F^{Aj}_{Bs} can be expressed in terms of the Kronecker deltas δ^a_b. For instance, for a second-rank tensor, when $A = (mn)$ and $B = (lr)$, we have

$$F^{Aj}_{Bs} = \delta^n_r \delta^m_s \delta^j_l + \delta^n_s \delta^m_l \delta^j_r.$$

For the variations of an element of four-dimensional infinitesimal volume dV_4 we have, in the general case, the following formulas

$$\begin{aligned}\delta\, dV_4 &= \left(\frac{\partial g}{2g} + \nabla_i \delta x^i \right) dV_4 \\ &= [(1/2)\, g^{ij}\, \partial g_{ij} + \nabla_i \delta x^i]\, dV_4,\end{aligned} \tag{2.5.2}$$

where

$$g = \det \| g_{ij} \|,$$

provided we vary the law of motion (2.1.1) and the metric of the space. For $\delta_d\, dV_4$ we have a similar formula with $\delta_d x^i$ substituted for δx^i.

In modeling space-time in varied four-dimensional Riemannian spaces, we can use the following determining formulas and formulas for the variation ∂:

$$R = g^{il} R_{il} = g^{il} R_{isl}{}^{s},$$

$$R_{ijl}{}^{s} = \partial \Gamma^s_{il} / \partial x^j - \partial \Gamma^s_{lj} / \partial x^i + \Gamma^s_{pj} \Gamma^p_{il} - \Gamma^s_{pi} \Gamma^p_{jl}.$$

As we know,

$$\partial R = -R^{ij}\, \partial g_{ij} + \nabla_l W^l, \quad \nabla_l W^l = g^{ij}\, \partial R_{ij},$$

$$W^l = (g^{ij} \delta^l_s - g^{il} \delta^i_s)\, \partial \Gamma^s_{ij} = (g^{ik} g^{lj} - g^{ij} g^{lk})\, \nabla_k\, \partial g_{ij}.$$

2.6 GIVEN SCALAR AND TENSOR PARAMETERS CHARACTERIZING MODELS OF MATERIAL MEDIA AND FIELDS

In the general case, the construction of macroscopic models of material media and fields is a thermodynamic problem whose solution may be built via the basic variational equation of the following type [6-17]:

$$\delta_d \int_{V_4} \Lambda\, dV_4 + \delta_d W^* + \delta_d W = 0, \tag{2.6.1}$$

where V_4 is arbitrary, and the scalar Lagrangian density Λ and the scalar functional $\delta_d W^*$, the latter being linear in the variations δ_d, must be defined on the basis of the first and second laws of thermodynamics and allowing for the specialization of thermodynamic functions as applied to the properties of the selected physical objects and the processes being studied. The term $\delta_d W$ is an integral over the surface Σ restricting the region of continuous values of the state characteristics and their variations in V_4. The integrand of the above-mentioned integral can be found from (2.6.1) [7-12, 15].

The various models of matter and fields and the properties of classes of phenomena can be characterized via a set of so-called determining parameters

$$\mu^A \quad (A = 1, 2, \ldots, N, \ldots),$$

some of which may be scalars while others may be components of vectors and tensors [2, 3, 14]. When applied to specific phenomena, the quantities μ^A must be defined from the solution of a given problem or from experiments as functions of coordinates and time ξ^i or x^i and the parameters

$$K^B(\xi^k),\ K^C(x^k)\quad (B = 1, 2, \ldots,\quad C = 1, 2, \ldots),$$

which are either physical constants or fixed functions of coordinates x^k or ξ^k. In particular, among the parameters μ^A there may be antisymmetric tensors defining intrinsic angular momenta, various vectors, tensor characteristics of dislocations, characteristics of electromagnetic and other fields, etc. The parameters K^B and K^C may be scalars or components of tensors of various ranks [2, 3, 14].

The compulsory presence of parameters K^B and K^C is due to a geometric selection of the observer's and comoving reference frames and a selection of the physical properties of the objects and processes considered. These are represented only by means of fixed parameters K^B and K^C in the definitions of thermodynamic functions, Λ and δW^*, in the particular systems under study and in the formulations of concrete problems.

As examples of parameters K^B and K^C we point to various quantities that may and must be present in the definitions of characteristic thermodynamic functions. These may be various kinds of constant scalars or scalar functions of time and position fixing the properties of the substances in question, such as the masses of characteristic individualized particle-volumes, tensor invariants (Young's and Poisson's moduli, quantities defining anisotropy, etc.) characterizing internal interactions in the system, concentrations of various chemical and phase components in mixtures (which must be given), constants in laws of diffusion, heat transfer, viscosity, radiation (or absorption) of light, and the like. The various substances and the properties of the various fields are distinguished only by K^B and K^C and the type of thermodynamic functions.

The parameters K^B and K^C may be the fixed or the variable in time and space tensor components. For instance, $K^C(x^k)$ may be the known components of the metric tensors $g_{ij}(x^k)$ in the observer's reference frame, while $K^B(\xi^i)$ may be the $\hat{g}_{ij}(\xi^\alpha, \tau_0)$ taken in the comoving reference frame only at $\tau = \tau_0$ but considered for any moment of time τ. These are either given or sought components of the metric tensor in the initial state. The corresponding tensors with constant components $K^B(\xi^\alpha)$ along world lines will change in the variable basis $\hat{\text{Э}}^i(\xi^\alpha, \xi^4)$ or $\tilde{\text{Э}}_i(\xi^\alpha, \xi^4)$. A tensor obtained in this manner can be used to define the tensor of finite deformations.

One can also introduce tensor components that fix the symmetry properties of the physical structure of small volumes of particles in a medium [2]; in some cases these are the tensor components for the density of intrinsic angular momenta, tensors fixing certain plastic deformations, or vectors of residual magnetization and polarization. The K^B may be the tensor components in the expressions for the internal energy, in equations of state for the energy-momentum tensor, in the electromagnetic equations of state related to the laws governing polarization and magnetization in an electromagnetic field; it may also be the constants that enter Ohm's law.

Introduction of the parameters K^B is necessary to provide a physical characteristic of the properties of the objects (i.e. models) being constructed that have to do with the objects' internal degrees of freedom and external interactions and, in general, all properties important for the class of phenomena considered.

By its very definition, the starting basic variational equation (2.6.1) for real phenomena that take place in an infinitesimal volume dV_4 tending to a point is an energy equation, in which we must exclude, prior to further steps, the external influx of heat $dQ^{(e)}$ entering dW^* via the second law of thermodynamics.

For varied phenomena Eq. (2.6.1) for finite volumes V_4 is not an energy equation for fixed masses. Here we must also exclude $\delta_d Q^{(e)}$ via the second law of thermodynamics generalized to varied states for each term in the integrand of $\delta_d W^*$.

For real phenomena we can write this equation as

$$d\int\limits_{V_4} \Lambda^0\, dV_4 + dW_0^* + dW = 0. \tag{2.6.2}$$

The differences $\Lambda - \Lambda^0$ and $\delta_d W^* - dW_0^* = \delta W^*$ must vanish when one goes over to real processes. This may occur automatically or on the basis of the Euler equations resulting from (2.6.1). For instance, in the presence of constraints this is achieved by applying the well-known Lagrange methods of multipliers.

The initial thermodynamic equation (2.6.2) may serve as a "leading" relationship for determining Λ and $\delta_d W^*$ in Eq. (2.6.1), which on the basis of the definition (2.4.3) of variations and Eq. (2.6.2) can be written thus:

$$\delta\int\limits_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0. \tag{2.6.3}$$

Obviously, the systems of Euler equations and of the equations of state for the basic equations (2.6.1) and (2.6.3) combined with Eq. (2.6.2) prove to be the same. In view of the Euler equations, a transition from variation δ_d in (2.6.1) to variation δ in (2.6.3) leads to equivalent relationships if $\Lambda - \Lambda^0 = 0$ and $dW^* - dW_0 = 0$.

Now let us study the question of varying the parameters K^B or K^C at $\delta\hat{\text{Э}}_i = 0$ or $\delta\text{Э}_i = 0$. Starting from the relations

$$\delta K^B(\xi^k) = \partial K^B(x^k) + \delta x^i \nabla_i K^B(x^k),$$
$$\delta_d K^B(\xi^k) = \partial K^B(x^k) + \delta_d x^i \nabla_i K^B(x^k),$$

we obtain

$$\delta K^B = 0, \quad \delta_d K^B = dx^i \nabla_i K^B, \quad \partial K^B = - \delta x^i \nabla_i K^B \tag{2.6.4}$$

when the $\hat{K}^B(\xi^k)$ are given and therefore not varied, or

$$\delta K^C = \delta x^i \nabla_i K^C, \quad \delta_d K^C = \delta_d x^i \nabla_i K^C . \tag{2.6.5}$$

when the $K^C(x^k)$ are given and therefore not varied.

If in the comoving system of coordinates the tensor components $K^B(\xi^k)$ are known (given or determined separately), in the observer's system of coordinates the tensor components $K^B(x^k)$ are generally not known, since $\xi^k = \xi^k(x^i)$ and the partial derivatives $\partial x^i/\partial \xi^l$ or $\partial \xi^s/\partial x^k$ connected with the law of motion of the medium and with transformation from the comoving reference frame to the observer's are the very functions that we are looking for. However, formulas (2.6.4) remain valid for variations of the function $K^B(x^k)$, i.e. variations that enter the arguments of the varied functions, and expressions of the type $\nabla_i K^B(x^k) = (\partial\xi^k/\partial x^i \ldots) \hat{\nabla}_k \hat{K}^B(\xi^s)$ contain derivatives of the sought functions, which enter the Euler equations in the observer's reference frame if the basic variational equation is used. If the $K^B(\xi^s)$ are functions only of the $\hat{g}_{pq}(\xi^k)$, then $\hat{\nabla}_k K^B(\xi^s) = 0$. Accounting for the fact that one condition of homogeneity and permanency of the properties of the medium or field, say

$$\nabla_i K^B(x^k) = 0 \quad \text{or} \quad \hat{\nabla}_i \hat{K}^B(\xi^k) = 0, \tag{2.6.6}$$

follows the other, we see that in this case Eq. (2.6.4) yields

$$\delta \hat{K}^B = \delta_d K^B = \delta K^B = 0. \tag{2.6.7}$$

Similar conclusions are true for K^C. In particular, Eqs. (2.6.6) and (2.6.7) are valid when either the $K^C = g_{ij}(x^k)$ or the $\hat{K}^B = \hat{g}_{ij}(\xi^k)$ are given.

In general relativity, the $g_{ij}(x^k)$ or $\hat{g}_{ij}(\xi^k)$ are the sought functions that must be varied. These functions obey the Euler equations and, hence, Eqs. (2.6.5) or (2.6.4) and, consequently, Eqs. (2.6.7) for g_{ij} or $\hat{g}_{ij}$ are not valid. Below we will show that application of the basic variational equation (2.6.3) provides a unified method for using the first and second laws of thermodynamics to find equations for physical problems in the form of Euler equations and also to find the subsidiary conditions on strong discontinuities and the boundary conditions, which contain all the conservation laws, all the dynamical equations, and all the equations for determining the characteristics of the internal degrees of freedom for the physical systems modeling the material media and the various fields.

On the other hand, some of the Euler equations and the corresponding conditions on strong discontinuities do not correspond to any physical conservation law. The attempts of some authors to always postulate and construct every equations for determining internal parameters as an analog of a conservation law cannot be considered justified. We must also note that constructing complex physical models of matter with internal degrees of freedom and of various physical fields by employing variational methods of a thermodynamic nature often proves to be the only way to solve such problems. For one, this is the case in the theory of simplified one- or two-dimensional problems introduced instead of complex three-dimensional problems.

The theory, developed below, of constructing models via the basic variational equation (2.6.3) can be considered as a far-reaching generalization of Lagrange's variational principle or as a generalization and further development of the virtual work principle for holonomic and nonholonomic systems and for reversible and irreversible processes.

2.7 THE DETERMINING PARAMETERS IN THE CHARACTERISTICS OF A CONTINUOUS MEDIUM AS A WHOLE AND THE CHARACTERISTICS OF INDIVIDUAL WORLD LINES

Every model of a medium or field is characterized by a set of variable or constant (sought or given) functions or parameters that must be introduced as arguments in characteristic thermodynamic functions, e.g. the specific internal energy U or the specific entropy S. These functions are usually postulated on the basis of actual or thought experiments or statistical theories applied to a specific type of macroscopic medium and phenomenon [6, 7]. We can also include in these characteristic quantities the locally or globally defined characteristics of space and time or other geometric and kinematic quantities connected with the law of motion of the given individualized point objects of a four-dimensional or three-dimensional nature.

In each set mentioned above, the characteristic quantities are the velocities of points, stress tensors and tensors defining the rate of deformation, rotation tensors, etc. Aside from this we must also introduce the scalar or tensor quantities μ^A , $A = 1, 2, 3, \ldots$, characterizing the physical and chemical states of particles and fields. These may include the various characteristics of the electromagnetic field, the concentration of various components, or the characteristics of various internal degrees of freedom, i.e. the internal angular momenta. Such characteristics may be scalars or tensors fixed, via physical considerations, by their components either in the observer's system of coordinates or in the comoving system for the motions being studied and the accompanying phenomena; we can also choose any other reference frame comoving with an appropriate observer [8-12, 16-18].

Some of the characteristic quantities similar to those just mentioned may be known beforehand, such as $K^B(\xi^i)$ or $K^C(x^k)$, or must be found in solving the problem. In particiular, this may be the law of motion $x^i(\xi^k)$ or the characteristics (included in the parameters μ^A) of the properties of electromagnetic processes given by functions of the electromagnetic field tensor components $\mathscr{F}_{ij}(x^k)$ and $H^{ij}(x^k)$, the magnetization and polarization tensor components $M_{ij}(x^k)$, and the components A_i of the vector potential.

Sometimes in defining thermodynamic functions we encounter cases where the arguments are components of tensors for which all or some of the indices refer to different bases, e.g. the observer's bases $\mathfrak{Э}_i$ and $\mathfrak{Э}^j$ or the bases in the comoving system of coordinates $\hat{\mathfrak{Э}}_i$ and $\hat{\mathfrak{Э}}^j$ and, in particular, the bases in the rest inertial reference frame. When calculating the physically essential three-dimensional acceleration, one has to find proper-time derivatives of three-dimensional velocity in the rest frame.

The covariance of all physical laws, a condition that follows automatically from the variational technique, leads to the necessity of varying in Eq. (2.6.3) any scalar functions of the Lagrangian density Λ that depend on arguments of a scalar or tensor nature. To obtain correct results, we must know all the arguments of both the sought tensor components and the tensor components given through parameters (the latter tensor components are either constant or variable in time and space).

We can say, on the basis of the general theory of tensor functions, that here the problem is the same as in establishing the π theorem in dimensionality theory, where we must include in the arguments of physically meaning functions all variable and constant parameters that have dimensions.

If we wish to employ the above theory to obtain a set of identities following from applying variations δ_Ω to scalar functions of Λ, we must vary in the arguments of Λ all tensor components not only of the unknown functions but of the given functions as well. In calculating $\delta_\Omega\Lambda$ we must consider all tensor components in one system of coordinates, e.g. in the observer's system of coordinates x^i, where, for one, we must introduce the following substitutions:

$$\hat{g}_{ij}\,(\xi^k) = g_{pq}\,(x^k) x^p{}_i x^q{}_j$$

or

$$\hat{T}^{\hat{i}}{}_{\hat{j}}{}^{k} = T^p{}_q{}^k\,(\partial \xi^{\hat{i}}/\partial x^p)\,(\partial x^q/\partial \xi^{\hat{j}})$$

(a hat over an index means that the index refers to the bases $\hat{\text{Э}}_i$ and $\hat{\text{Э}}^j$). As to the $x^i{}_{\hat{j}}$, these quantities can be considered as the components of four vectors

$$x^i{}_{\hat{j}}\text{Э}_i \quad (j = 1,\ 2,\ 3,\ 4).$$

Thus, when we reduce all tensors to the observer's reference frame, the dependence of Λ on the initial arguments $x^i{}_j$ may change. After we have reduced all tensors to one system of coordinates, we can apply δ_Ω to Λ along the lines developed in Sec. 2.2. We then obtain the appropriate relationships in the form of conditions according to which the coefficients of the $\Omega^m{}_n$ must vanish.

In constructing the variations $\delta_\Omega \int\limits_{V_4} \Lambda dV_4$ or simply $\delta_\Omega \Lambda$ we must also vary the tensor components K^B and K^C (g_{ij}, for one) in the same manner (discussed in detail in Sec. 2.2) as we would with components of any tensor in the arguments of scalar Λ. In building variational expressions containing the variations of arguments of the type δ (see Sec. 2.4) we need not reduce all tensors to one system of coordinates. In this case, to calculate the variations of the known functions of ξ^i or of x^i we may use Eqs. (2.6.4)-(2.6.7).

2.8 THE BASIC VARIATIONAL EQUATION AND IDENTITIES FOLLOWING FROM THE SCALAR NATURE OF THE LAGRANGIAN DENSITY

Let us take the basic variational equation [6, 7]

$$\delta \int\limits_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0. \tag{2.8.1}$$

Here Λ is the scalar Lagrangian density obtained via a mental generalization of thermodynamic functions on varied phenomena (Λ is generally the specific total energy of a particle but with a minus sign), the thermodynamic functions being defined for real phenomena related to the specific model, and dV_4 is the (scalar) volume element. The integration domain V_4 constitutes an arbitrary volume.

For the integrand in the first term on the left-hand side of (2.8.1) we can write

$$-\Delta\mathcal{E} = \Lambda dV_4 = \Lambda\,(dV_4/dV_{04})\;dV_{04} = \Lambda^0 dV_{04}, \tag{2.8.2}$$

where dV_4 is a liquid material four-dimensional volume of a particle in the comoving reference frame ξ^i, and dV_{04} is a similar volume in the observer's reference frame. We then have

$$\Lambda = -\Delta\mathcal{E}/dV_4, \quad \Lambda^0 = -\Delta\mathcal{E}/dV_{04}.$$

From the definition of the scalar quantity $\mathcal{E}$ related directly to energy and the concepts of dV_4 and dV_{04} as scalars it follows that Λ, Λ^0 and the ratio dV_4/dV_{04} are scalars.

In a small region the transition from dV_{04} to dV_4 is the result of an affine transformation, whereby the ratio dV_4/dV_{04} does not depend on the initial shape of dV_{04}.

Below we consider a fairly general model within the framework of special relativity[6] (the model is of a purely illustrative nature) in which Λ is given by the following formula:

$$\Lambda = \Lambda\,(x^i{}_j,\ \mu^A,\ \nabla_k\mu^A,\ S,\ g_{ij}\,(x^k),\ K^B\,(\xi^k),\ K^C\,(x^k)). \tag{2.8.3}$$

From (2.8.2) we see that Λ^0 depends on the same arguments as Λ but the dependence on $x^i{}_j$ is different. Since

$$\delta_\Omega\Lambda^0 = \delta_\Omega\,(\Lambda dV_4/dV_{04}) + \Lambda\delta_\Omega\,(dV_4/dV_{04}),$$

with $dV_4/dV_{04} \neq 0$, we find that

$$\delta_\Omega\,(dV_4/dV_{04}) = 0, \quad \delta_\Omega\Lambda^0 = 0, \quad \text{and} \quad \delta_\Omega\Lambda = 0. \tag{2.8.4}$$

According to the basic postulate, Eq. (2.8.1) in variations δ_d for an infinitesimal volume dV_4 is the equation of the first law of thermodynamics in which the external influx of heat is excluded via the second law of thermodynamics (this law is used on the assumption, for the sake of simplicity, that for small particle the concept of absolute temperature has a meaning).

We can write an expression for δW^* in accordance with the energy equation for real phenomena and the generalization of this equation that allow for additional external influences on an infinitesimal particle in varied processes and for the irreversible effects, which contribute to the influx of the uncompensated heat dQ' in the equation of the second law of thermodynamics

$$\rho T dS dV_4 = dQ^{(e)} + dQ', \tag{2.8.5}$$

also mentally generalized to varied processes.

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[6] More general models have been considered in [2,7]. In (2.8.3) we have assumed that the arguments can be taken in any reference frame of an observer but with a fixed comoving system ξ^i; for one, for the observer reference frame we can take a nonholonomic system of local (proper) inertial tetrads.

In the model (2.8.3) the expression for the scalar δW^*, which we must also postulate for varied processes, is taken in the form

$$\delta W^* = \int\limits_{V_4} (\rho T\delta S - Q_i^*\delta x^i - \tau_i{}^k\nabla_k\delta x^i + M_A\,\partial\mu^A)\,dV_4. \tag{2.8.6}$$

Here ρ is the density calculated with the mass at rest, T the temperature, $Q_i^* = Q_i + \mathscr{D}_i$, where Q_i is a component of the body external four-dimensional force and $\mathscr{D}_i$ a similar component of a force depending on the sought functions; $\tau_i{}^k$ and M_A are known or additionally sought functions of x^i or of the arguments in (2.8.3) or of other given or sought quantities—both are tensor components emerging from $\delta Q'$ and characterizing energy dissipation (for one, this may be the dissipation due to the mechanical or electromagnetic viscosity of the medium taken into account by $\tau_i{}^k$).

In the formula (2.8.6) for δW^* the last term in the integrand can be taken, without loss of generality, in the form $M_A\partial\mu^A$ when the coefficients M_A (zero or nonzero) are determined from the thermodynamics of irreversible phenomena. Obviously, the presence of terms in the integrand with Q_i^* and $\tau_i{}^k$, which must be determined from subsidiary conditions that allow for the possible presence of terms of the type $M_i\delta x^i$ and $M_i{}^k\nabla_k\delta x^i$ or terms of the type $M_A\delta\mu^A = M_A\,(\partial\mu^A + \delta x^i\nabla_i\mu^A)$ that might appear in justifying the models from the standpoint of physics, must be taken into consideration in additional studies with fixed expressions for Q_i^* and $\tau_i{}^k$.

In view of the scalar nature of the volume V_4, we find from (2.8.1) that the scalar δW is an integral over the three-dimensional surface Σ that is the boundary of V_4 and may contain two-sided surfaces of strong discontinuities inside V_4. In our example this integral is

$$\delta W = \int\limits_{\Sigma} (P_i^{\prime k}\delta x^i + N_A{}^k\,\partial\mu^A)\,n_k\,d\sigma$$

$$= \int\limits_{\Sigma} (P_i{}^k\delta x^i + N_A{}^k\delta\mu^A)\,n_k\,d\sigma. \tag{2.8.7}$$

Since Λ is a scalar and variations of tensors are tensors of the same nature, each term in the integrand of (2.8.7) is a scalar. Here the n_k are the components of a unit normal to Σ. As will be shown in the next section, the coefficients $P_i{}^k$ and $N_A{}^k$, which are tensor components, can be expressed via (2.8.1) in terms of the Lagrangian density Λ and the terms in δW^* that can be transformed into surface integrals over Σ.

The fact that

$$\delta_\Omega\Lambda = 0 \tag{2.8.8}$$

can be stated explicitly in the following way:

$$\delta_\Omega \Lambda = (\partial\Lambda/\partial x^i{}_j)\ \delta_\Omega x^i_j + (\partial\Lambda/\partial\mu^A)\delta_\Omega\mu^A + (\partial\Lambda/\partial\nabla_k\mu^A)\ \delta_\Omega\nabla_k\mu^A$$
$$+ (\partial\Lambda/\partial g_{ij})\delta_\Omega g_{ij} + (\partial\Lambda/\partial K^B)\delta_\Omega K^B$$
$$+ (\partial\Lambda/\partial K^C)\delta_\Omega K^C = 0. \tag{2.8.9}$$

where we have used the fact that S is a scalar. Equation (2.8.9) contains no terms with the variations δ_Ω of the scalar quantities from the set μ^A, K^B, and K^C.

To reduce the size of the formulas below that follow from (2.8.9) and yet retain their meaning (the formulas contain sums involving the tensor components μ^A, K^B, and K^C) and the typical features of (2.8.9), we leave only one term for each of these parameters. We take only one second-rank tensor with components $\mu^i{}_j$ instead of the tensors μ^A, one given vector with components $K^i\ (x^n)$ instead of the tensors K^C, and only one second-rank tensor with components $\hat{\varepsilon}_{is} = \hat{\varepsilon}_{pq}x^p{}_i x^q{}_s$ (e. g. the components of the tensor of remnant deformation, which are the characteristics of an individual particle whose position is given by the coordinates ξ^i) instead of the various tensors K^B. Using (2.8.3), (2.2.2), (2.2.3), we can rewrite Eq. (2.8.9) in the following form:

$$-x^p{}_j \frac{\partial\Lambda}{\partial x^i{}_j}\Omega^i{}_p - \frac{\partial\Lambda}{\partial\mu^i{}_j}\mu^p{}_j\Omega^i{}_p + \frac{\partial\Lambda}{\partial\mu^s{}_p}\mu^s{}_i\Omega^i{}_p$$
$$-\frac{\partial\Lambda}{\partial\nabla_k\mu^i{}_j}\nabla_k\mu^p{}_j\Omega^i{}_p + \frac{\partial\Lambda}{\partial\nabla_k\mu^s{}_p}\nabla_k\mu^s{}_i\Omega^i{}_p$$
$$+\frac{\partial\Lambda}{\partial\nabla_p\mu^s{}_k}\nabla_i\mu^s{}_k\Omega^i{}_p + 2\frac{\partial\Lambda}{\partial g_{pj}}g_{pj}\Omega^i{}_p - \frac{\partial\Lambda}{\partial K^i}K^p\Omega^i{}_p$$
$$+\frac{\partial\Lambda}{\partial\varepsilon_{pj}}\varepsilon_{ij}\Omega^i{}_p + \frac{\partial\Lambda}{\partial\varepsilon_{jp}}\varepsilon_{ji}\Omega^i{}_p = 0.$$

Since $\Omega^i{}_p$ is arbitrary, we have

$$-x^p{}_j\frac{\partial\Lambda}{\partial x^i{}_j} - \frac{\partial\Lambda}{\partial\mu^i{}_j}\mu^p{}_j + \frac{\partial\Lambda}{\partial\mu^s{}_p}\mu^s{}_i$$
$$-\frac{\partial\Lambda}{\partial\nabla_k\mu^i{}_j}\nabla_k\mu^p{}_j + \frac{\partial\Lambda}{\partial\nabla_k\mu^s{}_p}\nabla_k\mu^s{}_i$$
$$+\frac{\partial\Lambda}{\partial\nabla_p\mu^s{}_k}\nabla_i\mu^s{}_k - \frac{\partial\Lambda}{\partial K^i}K^p + \frac{\partial\Lambda}{\partial\varepsilon_{pj}}\varepsilon_{ij}$$
$$+\frac{\partial\Lambda}{\partial\varepsilon_{jp}}\varepsilon_{ji} = -2\frac{\partial\Lambda}{\partial g_{pj}}g_{ij}. \tag{2.8.10}$$

These are 16 identities (in i and p) obtained as a consequence of the scalar nature of the Lagrangian density Λ. After symmetrization and alternation they reduce to 10 symmetric and 6 antisymmetric tensor identities that do not contain the term with $\partial\Lambda/\partial g_{pj}$. Adding the same term $-\delta^p_j\Lambda$ to the right- and left-hand sides of (2.8.10) and allowing for the formulas

$$\delta^p_i = \frac{1}{|g_{ij}|}\frac{\partial|g_{ij}|}{\partial g_{pj}}g_{ij} = \frac{2}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g_{pj}}g_{ij}$$

and the relationships

$$-2\frac{\partial\Lambda}{\partial g_{pj}}g_{ij} - \delta^p_i\Lambda = -\frac{2}{\sqrt{-g}}\frac{\partial\Lambda\sqrt{-g}}{\partial g_{pj}}g_{ij},$$

that follow from these formulas, we can rewrite (2.8.10) in the following manner:

$$-x^p{}_j\frac{\partial\Lambda}{\partial x^i_j} - \Lambda\delta^p_i - \frac{\partial\Lambda}{\partial\mu^i{}_j}\mu^p{}_j + \frac{\partial\Lambda}{\partial\mu^s{}_p}\mu^s{}_i - \frac{\partial\Lambda}{\partial\nabla_k\mu^i{}_j}\nabla_k\mu^p{}_j$$

$$+\frac{\partial\Lambda}{\partial\nabla_k\mu^s{}_p}\nabla_k\mu^s{}_i + \frac{\partial\Lambda}{\partial\nabla_p\mu^s{}_k}\nabla_i\mu^s{}_k - \frac{\partial\Lambda}{\partial K^i}K^p$$

$$+\frac{\partial\Lambda}{\partial\varepsilon_{pj}}\varepsilon_{ij} + \frac{\partial\Lambda}{\partial\varepsilon_{jp}}\varepsilon_{ji} = -\frac{2}{\sqrt{-g}}\frac{\partial\Lambda\sqrt{-g}}{\partial g_{pj}}g_{ij}. \qquad (2.8.11)$$

The equivalent identities (2.8.10) and (2.8.11) can be used to transform the relationships that emerge later and to calculate in some important cases the energy-momentum tensor components.

2.9 THE EULER EQUATIONS FOR THE BASIC VARIATIONAL EQUATION (2.8.1)

Now let us apply the variation of the type δ to Eq. (2.8.1). Since

$$\delta\int_{V_4}\Lambda\,dV_4 = \int_{V_4}(\partial\Lambda + \delta x^i\nabla_i\Lambda + \Lambda\nabla_i\delta x^i)\,dV_4$$

$$= \int_{V_4}[\partial\Lambda + \nabla_i(\Lambda\delta x^i)]\,dV_4,$$

we find that for Λ defined by (2.8.3) and in view of the fact that $\partial g_{ij} = 0$, $\delta K^B = 0$ (because of this $\partial K^B = -\delta x^i\nabla_i K^B$), and $\partial x^i{}_j = x^s{}_j\nabla_s\delta x^i - \delta x^s\nabla_s x^i{}_j$, we arrive at the following formula (if we allow for (2.8.3) and (2.8.6)):

$$\int\limits_{V_4}\Big\{\Big[-\nabla_s\Big(\frac{\partial\Lambda}{\partial x^i{}_j}x^s{}_j\Big)-\frac{\partial\Lambda}{\partial x^s{}_j}\nabla_i x^s{}_j-\frac{\partial\Lambda}{\partial K^B}\nabla_i K^B-\frac{\partial\Lambda}{\partial S}\nabla_i S+\nabla_s\tau_i{}^s$$

$$-Q_i^*\Big]\delta x^i+\Big(\frac{\partial\Lambda}{\partial\mu^A}-\nabla_l\frac{\partial\Lambda}{\partial\nabla_l\mu^A}+M_A\Big)\partial\mu^A+\Big(\frac{\partial\Lambda}{\partial S}+\rho T\Big)\delta S$$

$$+\nabla_s\Big[\Big(\frac{\partial\Lambda}{\partial x^i_j}x^s{}_j+\Lambda\delta^s_i-\tau^s{}_i\Big)\delta x^i+\frac{\partial\Lambda}{\partial\nabla_s\mu^A}\partial\mu^A\Big]\Big\}dV_4+\delta W=0. \tag{2.9.1}$$

This together with the fact that δx^i, $\delta\mu^A$, and δS are arbitrary yields the following Euler equations:

at δx^i

$$-\nabla_s\Big(\frac{\partial\Lambda}{\partial x^i{}_j}x^s{}_j\Big)-\frac{\partial\Lambda}{\partial x^s{}_j}\nabla_i x^s{}_j$$

$$+\nabla_s\tau_i{}^s-\frac{\partial\Lambda}{\partial S}\nabla_i S-\frac{\partial\Lambda}{\partial K^B}\nabla_i K^B-Q_i-\mathscr{D}_i=0; \tag{2.9.2}$$

at $\partial\mu^A$

$$\frac{\partial\Lambda}{\partial\mu^A}-\nabla_s(\partial\Lambda/\partial\nabla_s\mu^A)+M_A=0; \tag{2.9.3}$$

at δS

$$\partial\Lambda/\partial S+\rho T=0. \tag{2.9.4}$$

From the divergence terms in (2.9.1) and the formula (2.8.7) for the surface integral δW we find that

$$P_i{}^k=-(\partial\Lambda/\partial x^i_j)x^k{}_j-\Lambda\delta^k_i+(\partial\Lambda/\partial\nabla_k\mu^A)\nabla_i\mu^A+\tau_i{}^k \tag{2.9.5}$$

since

$$N^k_A=-\partial\Lambda/\partial\nabla_k\mu^A. \tag{2.9.6}$$

On the basis of (2.9.5) we can write

$$\nabla_k P_i{}^k=-\nabla_k\Big(\frac{\partial\Lambda}{\partial x^i{}_j}x^k{}_j\Big)-\nabla_i\Lambda+\nabla_k\Big(\frac{\partial\Lambda}{\partial\nabla_k\mu^A}\nabla_i\mu^A\Big)+\nabla_k\tau_i{}^k,$$

and since

$$\nabla_i\Lambda=\frac{\partial\Lambda}{\partial x^k{}_j}\nabla_i x^k{}_j+\frac{\partial\Lambda}{\partial\mu^A}\nabla_i\mu^A$$

$$+\frac{\partial\Lambda}{\partial\nabla_k\mu^A}\nabla_i\nabla_k\mu^A+\frac{\partial\Lambda}{\partial S}\nabla_i S+\frac{\partial\Lambda}{\partial K^C}\nabla_i K^C+\frac{\partial\Lambda}{\partial K^B}\nabla_i K^B,$$

Eq. (2.9.2) combined with (2.9.3) yield

$$\nabla_k P_i{}^k=Q_i-(\partial\Lambda/\partial K^C)\nabla_i K^C+\mathscr{D}_i+M_A\nabla_i\mu^A. \tag{2.9.7}$$

If Eq. (2.9.7) reflects a conservation law, we must have

$$\mathscr{D}_i + M_A \nabla_i \mu^A = \nabla_k R_i{}^k, \tag{2.9.8}$$

where the tensor components $R_i{}^k$ are functions of the determining parameters. Condition (2.9.8) follows from the energy-momentum conservation law in the region of continuous motions and imposes certain restrictions on the choice of $\mathscr{D}_i$. In this case we can write Eq. (2.9.7) in the form

$$\nabla_k T_i{}^k = Q - (\partial\Lambda/\partial K^C)\nabla_i K^C, \tag{2.9.9}$$

where

$$\begin{aligned} T_i{}^k &= P_i{}^k - R_i{}^k \\ &= -x_j{}^k (\partial\Lambda/\partial x^i{}_j) - \Lambda\delta_i^k + \tau_i{}^k + [\partial\Lambda/\partial \; \nabla_k\mu^A)]\, \nabla_i\mu^A - R_i{}^k. \end{aligned} \tag{2.9.10}$$

This formula simplifies if

$$R_i{}^k = [\partial\Lambda/\partial\, (\nabla_k\mu^A)]\, \nabla_i\mu^A \quad \text{or} \quad R_i{}^k = [\partial\Lambda/\partial\, (\nabla_k\mu^{A^*})]\nabla_i\mu^{A^*}. \tag{2.9.11}$$

In the first formula in (2.9.11) the summation is over all the compound indices A, while in the second the summation is over the compound indices A^* chosen from all indices A.

In general, the tensors $\mathbf{P} = P_i{}^k \text{Э}^i \text{Э}_k$ and $\mathbf{T} = T_i{}^k \text{Э}^i \text{Э}_k$ are different; they coincide only when $R_i{}^k = 0$, for one, when $\mathscr{D}_i = 0$ and when the covariant derivatives $\nabla_i\mu^A$ of subsidiary parameters are not included in the parameters of Λ. Each of the two tensors can be called the energy-momentum tensor for the given model; the components $T_i{}^k$ enter the equation of conservation laws (2.9.9), while the components $P_i{}^k$, as we will show below, enter the conditions for conservation laws at strong discontinuities.

Equations (2.9.2)-(2.9.4) generally contain functions $\tau_i{}^s$ and M_A, which are needed for building the model and must be given separately as characteristics of irreversible effects. The components of the four-dimensional body force $Q_i + \mathscr{D}_i$ determine the interaction of an element of a given substance and field with neighboring elements and other objects. In the statement of a specific problem we must know how to fix such forces.

Specifying the functions $K^C(x^i)$ and $K^B(\xi^i)$, which are arguments in the functional definition of the Lagrangian density (2.8.3), is associated with a manifestation of certain thermodynamic properties of the model considered. For instance, we can take as the known parametric functions $K^C(x^i)$ the characteristics of an unperturbed electromagnetic field (fixed in the statement of the problem) or a given unperturbed gravitational field when we are considering the motion of matter for which the internal energy depends on the field characteristics. Another example is the motion of a liquid in a porous medium, for which the characteristic $K^C(x^i)$ of the frame deformation is given. For an individualized particle the function $K^B(\xi^i)$ may be the components of the remanent plastic deformation tensor or the remanent electrization and magnetization tensor; the values of these components in the particles must be constructed separately by a specially introduced theory or experimentally. On the other hand, these parameters may enter the expression for the internal energy and, hence, the Lagrangian density Λ.

The main equations (2.9.2)-(2.9.4), whose number is equal to that of the sought functions $x_j^{\ i} = \partial x^i/\partial \xi^j$, $\mu^A(x^k)$, and $S(x^k)$, form a closed system of partial differential equations.

All the equations derived above have a tensor nature. They can be considered in any reference frame of an observer, including the special frames related to defining the thermodynamic functions of the system combining the field and medium and also the comoving system of coordinates ξ^k. The reader is reminded that the aforementioned reference frames and tensors may be introduced both via holonomic systems of coordinates and via nonholonomic bases. For instance, all the tensor components and corresponding equations can be considered at each point of the four-dimensional space in local (proper) inertial reference bases $\overset{*}{\mathfrak{Э}}_i$ and $\overset{*}{\mathfrak{Э}}{}^k$.

In comoving or proper (rest) frames the parameters $K^C(x^k)$ will not be present in (2.8.3) if the presence of an arbitrary system of coordinates x^k of the observer is irrelevant. If because of certain physical interactions and corresponding special features a system of coordinates x^k of the observer plays an essential role, the parameters $K^C\,(x^k)$ calculated in the comoving or proper reference frame may prove to be significant. In such special cases the K^C will be included in the proper reference frames and thermodynamic functions together with $\overset{*}{x}{}^i{}_j$ and $x^i{}_j$ (where $\overset{*}{x}{}^i\,(\xi^k)$ are infinitesimal coordinates of individual points in a proper system of coordinates).

Formulas (2.8.11) imply that at $\mathscr{S}_i = 0$, $\tau_i^{\ j} = 0$, and in the absence of parameters μ^A, K^B, and K^C from the arguments of Λ the following formula for determining the energy-momentum tensor is true:

$$T_i^{\ k} = P_i^{\ k} = -\frac{2}{\sqrt{-g}}\,\frac{\partial \Lambda \sqrt{-g}}{\partial g_{kj}}\,g_{ij}. \tag{2.9.12}$$

This formula is common in many works and various applications but it is important to bear in mind that generally formulas (2.8.11) and (2.9.10) imply that (2.9.12) does not hold. However, it can be used in the more general cases, where the additional terms in (2.8.11) form combinations that vanish identically.

It is easy to see, and this is a well-known fact, that the set of equations (2.9.2), (2.9.3), and (2.9.4) remains the same for two Lagrangians Λ and Λ^* if

$$\Lambda^* = \Lambda + \nabla_i \Omega^i. \tag{2.9.13}$$

Here Ω^i are any functions of the determining parameters that are included in the arguments Λ according to (2.8.3); in general they may depend on any other arguments or simply be functions of the coordinates x^k.

The divergence term $\nabla_i \Omega^i$ influences only the equations of state (2.9.5) and (2.9.6) and, hence, the energy-momentum tensor. This term is a scalar when $\Omega^i \text{Э}_i$ is a vector.

To find how the formula for the energy-momentum tensor changes we will calculate the variation of this tensor when a term of the type $\nabla_i \Omega^i$ is added to Λ. We proceed with the calculations in the general case where $\partial g_{ij} \neq 0$, which is often the case in general relativity (here $g_{ij}(x^k)$ is a sought function [9, 10, 12]). We have

$$\delta \int\limits_{V_4} \nabla_i \Omega^i \, dV_4 = \int\limits_{V_4} \left[\partial \nabla_i \Omega^i + \frac{\partial \sqrt{-g}}{\sqrt{-g}} \nabla_i \Omega^i + \nabla_s (\nabla_i \Omega^i \delta x^s) \right] dV_4$$

$$= \int\limits_{V_4} \frac{\sqrt{-g}\, \partial \nabla_i \Omega^i + \nabla_i \Omega^i \partial \sqrt{-g}}{\sqrt{-g}} \, dV_4 + \int\limits_{\Sigma} \nabla_i \Omega^i \delta x^s n_s \, d\sigma.$$

To transform the first integral on the right-hand side we note that

$$\int\limits_{V_4} \frac{\partial (\sqrt{-g}\, \nabla_i \Omega^i)}{\sqrt{-g}} \, dV_4 = \int\limits_{V_4} \frac{1}{\sqrt{-g}}\, \partial \, \frac{\partial \sqrt{-g}\, \Omega^i}{\partial x^i} \, dV_4$$

$$= \int\limits_{V_4} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\partial \sqrt{-g}\, \Omega^i) \, dV_4$$

$$= \int\limits_{V_4} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left[\sqrt{-g} \left(\frac{\partial \sqrt{-g}\, \Omega^i}{\sqrt{-g}} \right) \right] dV_4$$

$$= \int\limits_{V_4} \nabla_i \frac{\partial \sqrt{-g}\, \Omega^i}{\sqrt{-g}} \, dV_4 = \int\limits_{\Sigma} \frac{\partial \sqrt{-g}\, \Omega^s}{\sqrt{-g}} n_s \, d\sigma.$$

Hence,

$$\delta \int\limits_{V_4} \nabla_i \Omega^i \, dV_4 = \int\limits_{\Sigma} \left[\nabla_i \Omega^i \delta x^s + \frac{1}{\sqrt{-g}} (\partial \sqrt{-g}\, \Omega^s) \right] n_s \, d\sigma = -\delta W_{\Omega^i}. \tag{2.9.14}$$

Within the framework of special relativity the formula (2.9.14) acquires a simpler form. Since $\partial\sqrt{-g} = 0$ we find that

$$\delta \int\limits_{V_4} \nabla_i \Omega^i \, dV_4 = \int\limits_{\Sigma} (\nabla_i \Omega^i \delta x^s + \partial \Omega^s) \, n_s \, d\sigma = -\delta W_{\Omega^i}.$$

Because of the additional term δW_{Ω^i} in δW the equations of state and the formulas (2.9.5) and, in general, (2.9.6) change.

The additional term $\nabla_i \Omega^i$ in the Lagrangian density occurs due to a physical assumption of a thermodynamic nature, since adding this term is equivalent to changing the thermodynamic functions for the varied states. Generally speaking, such a change is related to the addition of latent energy, which does not influence the physical equations (in particular, the equations of motion), i.e. the Euler equations, but can modify the equations of state and in this way influence the boundary conditions in boundary value problems; it can also manifest itself at strong discontinuities.

According to the general theory outlined here, the addition of the term $\nabla_i \Omega^i$ to the Lagrangian changes the tensor $P_i^{\ k}$ but leaves unchanged the tensor $T_i^{\ k}$ since the Euler equations retain their form.

2.10 THE CONDITIONS AT STRONG DISCONTINUITIES

As the reader may know, in many problems formulated in certain regions $\mathscr{D}$ of the four-dimensional space-time the only solutions are those with strong discontinuities on three-dimensional surfaces S_k inside $\mathscr{D}$ for the characteristic functions that are arguments in the thermodynamic function of the states of matter and field and in the Lagrangian density Λ (e.g. the quantities $x^i{}_j$, μ^A , and S may have discontinuities).

The sought functions that have discontinuities must satisfy certain conditions on the surface with the discontinuities. In the simplest cases such conditions can be obtained via universal conservation laws that allow for possible external interactions (the lifting surface and the pull in a propeller, and the like). The conditions may also emerge as a result of various assumptions introduced in the regions of continuous smooth variations of the characteristics of the introduced model, which is built with no regard for various important properties and effects inherent in more complex models, properties and effects that manifest themselves and must be taken into account by introducing sudden jumps in the simpler model within regions where the sought quantities have large gradients (e.g. viscous properties, chemical reactions in combustion and detonation, absorption or liberation of energy in phase transitions, surface currents, etc.).

For models with internal degrees of freedom and additional characteristic parameters of the type μ^A , if we are to obtain the conditions at discontinuities we must rely on thermodynamic laws of the type of the energy conservation law. Such regularities must be included in the basic equation (2.8.1). For this, if there are external interactions of the type of distributed surface forces or moments on, say, the lifting surfaces, we must introduce additional fixed integrals over the surface of discontinuity S_k into the expression for δW^*.

In the absence of external interactions, Eq. (2.8.1) with Λ defined by (2.8.3) yields

$$\int\limits_{S_1+S_2+\dots} \{[P_i{}^k \delta x^i n_k \, d\sigma] + [M_A^k \delta \mu^A n_k \, d\sigma]\} = 0 \qquad (2.10.1)$$

after we have derived the Euler equations and the equations of state from the expression for δW. The surface integral in (2.10.1) encompasses both sides of all discontinuity surfaces $S_1, S_2, \dots, S_m$. Since the basic variational equation is formulated on the assumption that all variations of the sought functions δx^i and $\delta \mu^A$ inside V_4, on Σ, and on S_k are continuous and arbitrary, it follows that at each point S_k the following relations must be true:

$$(P_{1i}{}^k \, d\sigma_1 - P_{2i}{}^k \, d\sigma_2)\, n_k = 0, \quad (M_{1A}^{\;k} \, d\sigma_1 - M_{2A}^{\;k} \, d\sigma_2) n_k = 0. \quad (2.10.2)$$

Here the indices 1 and 2 denote the two sides of the discontinuity surface S_k, and we have taken into account that $\mathbf{n}_{(1)} = -\mathbf{n}_{(2)} = \mathbf{n}$. In addition, we assume that, generally speaking, on both sides of the surface S_k the metric may also undergo a discontinuity and hence $d\sigma_1 \neq d\sigma_2$, which may be the case in general relativity.

If on the discontinuity surfaces S_k the metric is continuous, $d\sigma_1 = d\sigma_2$. In this case we can rewrite condition (2.10.2) in the following form:

$$P_{1i}{}^k n_k - P_{2i}{}^k n_k = 0, \qquad (2.10.3)$$

$$M_{1A}^{\;k} n_k - M_{2A}^{\;k} n_k = 0. \qquad (2.10.4)$$

Equation (2.10.3) can be simplified by applying Eq. (2.10.4).

If there are external influences, we must place in the right-hand sides of (2.10.3) and (2.10.4) the quantities that characterize these influences. These terms will be in the form of surface integrals to be inserted into Eq. (2.8.1) and similar to δW^*. Such integrals must be introduced additionally on the basis of the physical data on influx or outflow of external energy at the discontinuities and in the same way as for δW^* in continuous motion. If the motion and the corresponding parameters μ^A at the discontinuities are given, then expressions of the type (2.10.3) and (2.10.4) but with nonzero right-hand sides can be used to find the values of the external discontinuity influences.

If the arguments of Λ include derivatives of the sought functions μ^A and x^i with an order higher than the first, the integrand in δW

will have terms of the type

$$M_A^{ik}\nabla_i\delta\mu^A n_k,\quad M_A^{ijk}\nabla_i\nabla_j\delta\mu^A n_k,\quad N_i^{jk}\nabla_j\delta x^i n_k,\quad N_i^{ljk}\nabla_l\nabla_j\delta x^i n_k,\ \text{etc.}$$

In this case the integrand in δW can be transformed in a one-to-one manner[7] so that the result will contain only covariant derivatives of $\delta\mu^A$ and δx^i of various orders along the normal n to S_k, e.g. only terms of the type

$$M'_A\nabla_n\delta\mu^A,\quad M''_A\nabla_n\nabla_n\delta\mu^A,\quad N'_i\nabla_n\delta x^i,\quad N''_i\nabla_n\nabla_n\delta x^i,\ \text{etc.}$$

The coefficients of δx^i and $\delta\mu^A$ change as a result of such transformations. We will denote these coefficients by $\widetilde{P}_i$ and $\widetilde{M}_A$, respectively.

After such a transformation the conditions at discontinuities take the form

$$\begin{aligned}
&\widetilde{P}_{i1}+\widetilde{P}_{i2}=0, && \widetilde{M}_{A1}+\widetilde{M}_{A2}=0,\\
&N'_{i1}-N'_{i2}=0, && M'_{A1}-M'_{A2}=0, \qquad (2.10.5)\\
&N''_{i1}+N''_{i2}=0, && M''_{A1}+M''_{A2}=0,
\end{aligned}$$

where we have allowed for the continuity of δx^i, $\nabla_n\delta x^i$, $\nabla_n\nabla_n\delta x^i$, $\nabla_n\delta\mu^A$, and $\nabla_n\nabla_n\delta\mu^A$ and the sign convention for the derivatives ∇_n and $\nabla_n\nabla_n$ on the two sides of each S_k. We may arrive at similar relationships when the arguments contain derivatives of a higher order. It is obvious that when the arguments of Λ contain higher-order derivatives of the sought functions, the order of the Euler equations increases.

We note that the conditions at the boundaries of the region of continuous solutions may be obtained as conditions at discontinuities, but we must either postulate or know the values of the characteristic quantities in the exterior of the boundaries. It is also clear that the conditions at the discontinuities (2.10.3)-(2.10.5) while retaining their general form change significantly when a term of the type $\nabla_i\Omega^i$ is added to the Lagrangian density; however, the Euler equations remain unchanged while the boundary conditions and the conditions at the discontinuities do change [10]. Here we have explicitly stated the fact (often ignored by authors) that the conditions at discontinuities and the boundary conditions cannot be derived from the equations of motion.

The building of models of matter and fields is always closely linked to the problem of universal postulates and the particular variants of the postulates that define a model. Among the universal postulates used in everyday life are the concepts of space and time, the concepts of various types of macroscopic and microscopic energies, which transform into each other, and the first and second laws of thermodynamics. An important assumption is the scalar nature of the local energy equations. This assumption leads to the conclusion that Λ and δW^* are scalars, which, in turn, guarantees the covariance of the complete system of the Euler equations and the auxiliary conditions in specific problems. Notwithstanding the covariance, physical effects depend generally on the observer's reference frame and the physical nature of the phenomena studied. Experiments and the impressions of the observer depend not only on the object observed and the processes occurring in the object but on the choice of the observer's rest frame (the reference frame comoving with the observer).

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[7] For details see [2, 7].

From the covariant nature of the formulation of the problem there follows, for one, the invariance of all determining equations with respect to particular coordinate transformations (the transition from one observer to another) that retain the invariance of the metric tensor components. This is the basis on which one formulates the Galilean transformation laws in Newtonian physics and the Lorentz transformation laws of relativistic physics concerning the equivalence of various reference frames.

As already noted, a system of none but Euler equations can be obtained for various Λ. But the chosen system of equations, boundary conditions and conditions at discontinuities obtained from the basic variational equation by fixing Λ and δW^* (in the absence of additional constraints) markedly reduces the degree of arbitrariness in choosing Λ and δW^*. This problem is similar to the one that we encounter when we look for the quantities in the Newtonian equation

$$m\mathbf{a} = \mathbf{F}.$$

But how is one to select the Λ and δW^*? In many works this is done in such a way that the Euler equations and the conditions at the discontinuities coincide with those of well-known models or with equations that the author expects to obtain. But obviously, such a method is expedient only in perfecting the technique of operating with the basic variational equation and in deepening the intuitive aspect in formulating new problems.

Specific postulates and assumptions in the development of new models must take into account the known data of a thermodynamic nature and must firmly rest on the postulated or separately established physical properties of the objects and the mechanisms of the internal and external interactions. Statistical theories may also prove to be expedient in the choice of Λ and δW^*.

2.11 ON MODELS OF FLUIDS

The mass m and the energy $\mathscr{E}$ are characteristics of matter and fundamental physical concepts, which for both macroscopic and microscopic objects are introduced as scalars. In macroscopic Newtonian mechanics of individual particles these are three-dimensional scalars. In the special and general theories of relativity we can introduce in an appropriate way the notions of rest mass m_0 and energy $\mathscr{E}$ as four-dimensional scalars.

The concepts of mass and energy are closely related to the equations where these concepts appear and which are built for a model description of the physical phenomena. On the basis of such equations, which enable interpreting, predicting, and "understanding" the observed effects, we are able to form intuitive and practical ideas concerning the essence of mass and energy.

The physical meaning of the nature of the macroscopic concepts of mass and energy may be interpreted in various ways. For instance, for a small particle of a material medium we can introduce the notion of the state of rest, i.e. consider the particle and its physical state as being in the comoving and, respectively, local rest frame of reference and introduce for such reference frames the rest mass dm_0, the internal energy dU^*, and the geometrically individualized spatial three-dimensional volume dV_3.

The internal energy dU^*, being a characteristic of a macroscopic state, emerges as a result of microscopic internal interactions, macroscopic internal thermal or other types of motion, or some postulated physical properties of material medium, field, of four-dimensional space-time. Formulas for dU^* in the classical theory of a small particle at rest may be introduced in terms of the determining parameters with due regard for internal interactions of a quantum nature.

As a universal relationship (postulate) whose validity can be proved in a number of special theories we take the following relationship between mass and energy:

$$d\mathscr{E} = dmc^2 = dm_0c^2 + dU^*. \tag{2.11.1}$$

Here c^2 is the square of the speed of light, and dm is the elementary mass defined in terms of dm_0, which is the elementary mechanical mass similar to the mass of a material point and remaining the same during internal processes in the particle (although dm_0 may change in nuclear reactions and in absorption or emission of some types of energy). In many theories and in practical life we may consider dm_0 to be a constant characteristic of a given individualized particle. For a continuous medium, dU^* is the macroscopic thermodynamic energy of an individualized particle determined primarily by microscopic thermal motion and internal forces or generally quantum interaction connected with deformations, internal structure, and various internal processes in the small particle (chemical reactions, phase transitions, etc.). The quantity dU^* may be also connected with various fields, primarily of an electromagnetic nature, and may also change in absorption and emission of various types of energy.

Alongside the notions of infinitesimal masses and energies for small particles and small volumes of fields, it is often expedient to introduce the concept of their densities, which are finite ratios of the mass and energy of an individualized particle to its three-dimensional volume, which changes along world lines.

Let us now see how the three-dimensional volume of an individualized small particle changes along the world line of the particle in the four-dimensional space-time with a metric. In the Minkowski space-time for any coordinate system x^i of the observer and the comoving system of coordinates ξ^i we have

$$ds^2 = g_{ij}\,dx^i\,dx^j = \hat{g}_{ij}\,d\xi^i\,d\xi^j = \hat{g}_{44}\,(d\xi_1^4)^2 + \hat{h}_{\alpha\beta}\,d\xi^\alpha\,d\xi^\beta,$$

$$d\xi_1^4 = d\xi^4 + \hat{g}_{4\alpha}\,d\xi^\alpha/\hat{g}_{44},$$

with[8]

$$\hat{h}_{\alpha\beta} = \hat{g}_{\alpha\beta} - \hat{g}_{\alpha4}\hat{g}_{\beta4}/\hat{g}_{44} = \hat{g}_{\alpha\beta} - \hat{u}_\alpha\hat{u}_\beta, \qquad dl^2 = -\hat{h}_{\alpha\beta}\,d\xi^\alpha\,d\xi^\beta,$$

where dl is the length element in a space that is orthogonal to the given world length at a point M.

Let dV_4 be an infinitesimal four-dimensional volume, while dV_3 is the common three-dimensional spatial volume linked to the former by the following relationships:

$$\begin{aligned} dV_3 &= -\sqrt{|\hat{h}_{\alpha\beta}|}\,d\xi^1\,d\xi^2\,d\xi^3; \\ dV_4 &= \sqrt{-\hat{g}}\,d\xi^1\,d\xi^2\,d\xi^3\,d\xi^4 = dV_3\sqrt{\hat{g}_{44}}\,d\xi^4 = dV_3\,ds. \end{aligned} \tag{2.11.2}$$

Here $\hat{g} = \det\|\hat{g}_{ij}\|$, and ds is the world line element at an arbitrary point M. We can write similar relationships for another point, say M_0, on the same world line $\xi^\alpha = \text{const}$ with the same values of $d\xi^\alpha$ and $d\xi^4$ but in different $\hat{Э}_i$ and $\hat{Э}_i^0$. Obviously,

$$dV_4/dV_{04} = \sqrt{\hat{g}/\mathring{g}} = (dV_3/dV_{03})\sqrt{\hat{g}_{44}/\mathring{g}_{44}},$$

whence

$$dV_3 = dV_{03}\sqrt{(\mathring{g}_{44}/\hat{g}_{44})\,(\hat{g}/\mathring{g})}. \tag{2.11.3}$$

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[8] The reader can easily verify that when the comoving coordinate system undergoes the following transformation:

$$\xi'^4 = f(\xi^4, \xi^\alpha), \quad \xi'^\alpha = \xi^\alpha,$$

the tensor components $h_{\alpha\beta}$ remain invariant although the metric tensor components $\hat{g}_{ij}$, e.g. $\hat{g}_{44}$, do not.

Now let us define the densities. We can either introduce the density ρ by the formula

$$\rho = dm_0/dV_3 \tag{2.11.4}$$

or, according to (2.11.1), the density ρ^* by the formula

$$\rho^* = dm/dV_3 = [dm_0 + (1/c^2)\, dU^*]/dV_3. \tag{2.11.5}$$

Definition (2.11.4) is useful when the rest mass dm_0 is constant along a world line, which is a typical case in many applications and phenomena. If we use definition (2.11.5), we can see that dm of an individual particle varies. If dU^* is an infinitesimal of the order dV_3, the second term in (2.11.5) is an infinitesimal of the order $1/c^2$; to within this precision definitions (2.11.4) and (2.11.5) coincide. Combining (2.11.4) with (2.11.3), we find that

$$\rho = \frac{dm_{01}}{dV_3} = \frac{dm_{01}}{dm_0}\,\frac{dm_0}{dV_{03}}\,\sqrt{\overset{\circ}{g}\,\overset{\circ}{g}_{44}}\,\sqrt{\frac{g_{pq}\,(\partial x^p/\partial\xi^4)\,(\partial x^q/\partial\xi^4)}{|g_{pq}\,(\partial x^p/\partial\xi^i)\,(\partial x^q/\partial\xi^j)|}},$$

where dm_{01} is the mass element of a particle at point M, and dm_0 is the mass element at point M_0, or

$$\rho = \frac{dm_{01}}{dm_0}\,\rho_0\,(\xi^\alpha,\ \xi_0^4)\,\sqrt{\overset{\circ}{g}\,\overset{\circ}{g}_{44}}\,\frac{\sqrt{g_{pq}x_4^p x_4^q}}{|x_j^i|\,\sqrt{|g_{pq}|}}. \tag{2.11.6}$$

The law of rest mass conservation can be written in the following way:

$$dm_{01} = dm_0. \tag{2.11.6$'$}$$

Equation (2.11.6) is a four-dimensional relativistic equation of continuity for finite deformations, when the ratio dm_{01}/dm_0 is known.

When dm_{01}/dm_0 does not depend on $x^i{}_j$, formula (2.11.6) yields

$$x^p{}_j\,(\partial\rho/\partial x^i{}_j) = -\,\rho\,(\delta_i^p - u_i u^p), \tag{2.11.7}$$

where $u^p = dx^p/ds$ and $u_i = g_{iq}\,(dx^q/ds)$. Indeed, Eq. (2.11.7) can be obtained directly by the following line of relationships:

$$x^p{}_j\,\frac{\partial\,|x^r{}_s|}{\partial x^i{}_j} = |x^r{}_s|\,\delta_i^p, \qquad x^p{}_j\,\frac{\partial\sqrt{g_{rs}x_4^r x_4^s}}{\partial x^i{}_j} = x^p{}_4\,\frac{g_{is}x^s{}_4}{\sqrt{g_{rs}x_4^r x_4^s}}$$

$$= \sqrt{g_{rs}x_4^r x_4^s}\,\frac{x_4^p g_{is}x_4^s\,(d\xi^4)^2}{ds^2} = \sqrt{g_{rs}x_4^r x_4^s}\,u_i u^p. \tag{2.11.8}$$

Now let us turn to the problem of constructing models of fluids. The basic feature by which fluids differ from other models of deformable bodies is the assumption that the dependence of the internal and the total energy of the deformation of the particles manifests itself solely in the variation of the density ρ defined by (2.11.4). One also assumes that the arguments $x^i{}_j$ enter Λ only through ρ. In other words, in typical examples of fluids we take the following postulate:

$$\Lambda = -[\rho c^2 + \rho U(\rho, S, \mu^A, \nabla_k\mu^A, g_{ij}(x^k), K^C(x^k))], \qquad (2.11.9)$$

where U is the internal energy per unit rest mass, μ^A the additional sought (variable) parameters (which may be either scalars or components of tensors), and S the specific entropy.

For reversible processes, i.e. processes in ideal fluids, we have, by definition,

$$\rho T\ dS = dQ^{(e)}.$$

Whence, in the absence of external body forces we assume that

$$\delta W^* = \int_{V_4} \rho T\,\delta S\,dV_4. \qquad (2.11.10)$$

But in some cases, which will be discussed later, e.g. a fluid in an electromagnetic field, the expressions (2.11.9) and (2.11.10) for Λ and δW^* become more complex even in the absence of Joule's heat.

In irreversible processes the expression for δW^* is generally more complex. For instance, in some cases we must assume the basis of the thermodynamics of irreversible processes that

$$\delta W^* = \int_{V_4} (\rho T\,\delta S - \tau_i{}^k\nabla_k\delta x^i + M_A\,\partial\mu^A - Q_i\delta x^i)\,dV_4. \qquad (2.11.11)$$

At $\mathscr{D}_i = 0$ the main problem in defining Q_i, $\tau_i{}^k$, and M_A lies in the irreversibility of the phenomena. This is a special rheological problem of thermodynamics.

On the basis of formula (2.9.5) and the notation $P = \rho^2\,(\partial U/\partial\rho)$ we can write

$$P_i{}^k = -P\,(\delta_i^k - u_iu^k) + \rho\,(c^2 + U)\,u_iu^k + \tau_i{}^k$$
$$-\rho\,(\partial U/\partial\nabla_k\mu^A)\nabla_i\mu^A, \qquad (2.11.12)$$

since

$$-N_A^k = \partial\Lambda/\partial\nabla_k\mu^A = \rho\partial U/\partial\nabla_k\mu^A$$

on the basis of (2.9.5), (2.9.6), and (2.11.9). All the Euler equations and the general form of the state equations are given in Sec. 2.9.

If we apply the variation δ_Ω to this case, we immediately notice that the density ρ and the entropy S are scalars and therefore $\delta_\Omega\,\rho = 0$ and $\delta_\Omega\,S = 0$. The corresponding nontrivial identities (2.8.10) and (2.8.11) emerge only because of the presence of tensor components in (2.11.9).

2.12 AN ELASTIC-BODY MODEL

When builiding models of deformable "solid" bodies, the main characteristic of deformation is the tensor of finite deformation, which is defined in a local comoving reference frame through spatial variations of infinitesimal individual segments of length dl (perpendicular at each point to the world line passing through this point) along a world line and corresponding to infinitesimal increments of the Lagrangian coordinates $d\xi^1$, $d\xi^2$, $d\xi^3$.

Thus, according to Sec. 2.10, we can introduce the length of the above-mentioned segment according to the formula

$$dl^2 = -\hat{h}_{\alpha\beta} d\xi^\alpha d\xi^\beta = -\hat{h}_{ij} d\xi^i d\xi^j, \tag{2.12.1}$$

where we have taken into consideration that $\hat{h}_{4j} = 0$.

The simplest and at the same time most widely used characteristic of the length of a "fluid" segment is the difference

$$dl^2 - dl_0^2 = (\hat{\mathring{h}}_{\alpha\beta} - \hat{h}_{\alpha\beta})\, d\xi^\alpha d\xi^\beta,$$

where $\hat{\mathring{h}}_{\alpha\beta} = \hat{\mathring{h}}_{\alpha\beta}(\xi^\alpha, \xi_0^4)$ and $\hat{h}_{\alpha\beta} = \hat{h}_{\alpha\beta}(\xi^\alpha, \xi^4)$; correspondingly, for each two points M_0 and M on the same world line there are two different tensors:

$$E = \hat{E}_{ij} \hat{\text{Э}}^i \hat{\text{Э}}^j = (1/2)\, (\hat{\mathring{h}}_{ij} - \hat{h}_{ij})\, \hat{\text{Э}}^i \hat{\text{Э}}^j$$

and

$$\mathring{E} = \hat{E}_{ij} \mathring{\text{Э}}^i \mathring{\text{Э}}^j = (1/2)\, (\hat{\mathring{h}}_{ij} - \hat{h}_{ij})\, \mathring{\text{Э}}^i \mathring{\text{Э}}^j, \tag{2.12.2}$$

which can be taken at point M only if we bear in mind that the tensor components $\hat{\mathring{h}}_{ij}(\xi^\alpha)$ are constant along a given world line, $\hat{\text{Э}}^i$ are the basic vectors at the running points M, and $\mathring{\text{Э}}^i$ are the basis vectors at point M_0, while at point M these vectors ($\mathring{\text{Э}}^i$) are obtained by their translation from point M to point M_0 along the world line connecting the two points. In special relativity, obviously, this translation does not depend on the path.

In what follows we will deal solely with tensor E. For its covariant components in the comoving reference frame we have

$$\hat{E}_{pq} = (1/2)[\hat{g}_{pq}(M_0) - \hat{u}_p(M_0)\hat{u}_q(M_0) - \hat{g}_{pq}(M) + \hat{u}_p(M)\hat{u}_q(M)]. \tag{2.12.3}$$

In the observer's reference frame, x^i, the contravariant component of this tensor are

$$E^{pq} = (1/2)\left[\overset{\circ}{\hat{h}}_{rs}\xi^r_m\xi^s_n g^{mp}g^{nq} - g^{pq} + \frac{\partial x^p}{\partial \xi^4}\frac{\partial x^q}{\partial \xi^4}\frac{1}{g_{rs}(\partial x^r/\partial \xi^4)(\partial x^s/\partial \xi^4)}\right]$$

$$(\xi^l_d = \partial\xi^l/\partial x^d). \qquad (2.12.4)$$

Let us consider the definition for the Lagrangian density for an elastic medium, a definition which generalizes that given by formula (2.11.9):

$$\Lambda = -[\rho c^2 + \rho U(E^{pq},\ S,\ \mu^A,\ \nabla_k\mu^A,\ g_{ij}(x^k),\ K^C(x^k))]. \qquad (2.12.5)$$

In contrast to (2.11.9), the arguments of U include the tensor components E^{pq} instead of ρ; these components are expressed in terms of $x^i{}_j$, $g^{ij}(x^k)$, and $\overset{\circ}{\hat{h}}_{pq}(\xi^\alpha,\ \xi^4_0)$ via formula (2.12.4). Using this formula, we apply the general theory developed in the preceding sections. For the sake of simplicity, in the arguments in (2.12.5) we exclude additional arguments of the type K^B, e.g. tensor components that fix the three-dimensional symmetry in the "initial state" spaces at different points of the medium. In addition to the general definition (2.12.5) for an elastic-body model with possible internal degrees of freedom represented by parameters μ^A, we note that the characteristic feature of an elastic medium is the reversibility of processes associated with the medium's motion and the internal transmutations described by changes in the parameters μ^A.

By definition, the second law of thermodynamics for an elastic medium and for real and varied phenomena is represented by the following equations:

$$\rho T\,dS = dQ^{(e)} \text{ and } \rho T\delta S = \delta Q^{(e)}. \qquad (2.12.6)$$

Hence, we take the main postulate for δW^* in the case of an elastic body in the following form:

$$\delta W^* = \int_{V_4}(\rho T\delta S - Q_i\delta x^i)\,dV_4, \qquad (2.12.7)$$

where, as before, Q_i are the components of the external body four-dimensional force ($\mathscr{D}_i = 0$).

We will now calculate $p_i{}^k$ via (2.9.5) at $\tau_i{}^k = 0$. Bearing in mind (2.11.12), we have

$$p_i{}^k = \rho(c^2 + U)u_iu^k + \rho\frac{\partial U}{\partial E^{pq}}x^k_i\frac{\partial E^{pq}}{\partial x^i{}_j} + \rho\frac{\partial U}{\partial \nabla_k\mu^A}\nabla_i\mu^A. \qquad (2.12.8)$$

To calculate $x^k{}_j(\partial E^{pq}/\partial x^i{}_j)$ on the basis of (2.12.4), we note the following auxiliary relationships: since $\xi^p_q x^q_n = \delta^p_n$ and, hence, $\xi^p_q dx^q_n + x^q_n d\xi^p_q = 0$, multiplying the latter relationship by ξ^n_m and contracting the product with respect to n, we obtain

$$d\xi^p_m = -\xi^n_m \xi^p_q \, dx^q_n, \text{ whence } \partial\xi^p_m/\partial x^q_n = -\xi^n_m \xi^p_q. \tag{2.12.9}$$

Combining (2.12.9) with (2.12.4), we obtain

$$x^k{}_j \partial E^{pq}/\partial x^i{}_j = (1/2)\,[-\hat{\mathring{h}}_{rl}\xi^r_i \xi^l_n g^{kp} g^{nq} - \hat{\mathring{h}}_{rl}\xi^r_m \xi^l_i g^{mp} g^{kq} + (\delta^p_i x^k_4 x^q_4 + \delta^q_i x^k_4 x^p_4)\,(d\xi^4)^2/ds^2 - 2u^p u^q u^k u_i]. \tag{2.12.10}$$

After substituting (2.12.10) into (2.12.8) we must contract the result with respect to p and q; since $\partial U/\partial E^{pq}$ is symmetric in p and q, we find

$$p'_i{}^k = \rho\,(c^2 + U)\, u_i u^k + \rho\,(\partial U/\partial E^{pq})\,(-\mathring{h}_i{}^q g^{kp} + \delta^p_i u^q u^k - u^p u^q u^k u_i), \tag{2.12.11}$$

where we have dropped the last term in (2.12.8). Here the tensor components $\mathring{h}_i{}^q$ are calculated in the observer's reference frame and are given by the formula

$$\mathring{h}_i{}^q = \hat{\mathring{h}}_{rs}\xi^r_i \xi^s_n g^{nq}. \tag{2.12.12}$$

Equation (2.12.11) is the generalized state equation for four-dimensional strains in nonlinear elasticity theory within the framework of special relativity. The corresponding state equations for the three-dimensional Newtonian elasticity theory may be found in [1].

In the comoving reference frame, $\hat{u}^4 = d\xi^4/ds = 1/\sqrt{\hat{g}_{44}}$, $\hat{u}^\alpha = 0$, and $\hat{u}_i = \hat{g}_{i4}/\sqrt{\hat{g}_{44}}$. Whence

$$\hat{p}'_4{}^4 = \rho\,(c^2 + U), \quad \hat{p}'_\alpha{}^\beta = \rho \frac{\partial U}{\partial \hat{E}^{pq}} (\hat{\mathring{h}}_\alpha{}^q g^{\beta p}),$$

$$\hat{p}'_\alpha{}^4 = \rho\,(c^2 + U)\,\frac{\hat{g}_{\alpha 4}}{\hat{g}_{44}} + \rho \frac{\partial U}{\partial \hat{E}^{pq}} \left(-\hat{\mathring{h}}_\alpha{}^q \hat{g}^{4p} + \delta^p_\alpha \delta^q_4 \frac{1}{\hat{g}_{44}} - \delta^p_4 \delta^q_4 \frac{\hat{g}_{\alpha 4}}{\hat{g}^2_{44}} \right)$$

$$= \rho\,(c^2 + U)\,\frac{\hat{g}_{\alpha 4}}{\hat{g}_{44}} + \rho \frac{\partial U}{\partial \hat{E}^{pq}} (-\hat{\mathring{h}}_\alpha{}^q g^{4p}) + \rho \frac{\partial U}{\partial \hat{E}^{\alpha 4}} \frac{1}{\hat{g}_{44}} - \rho \frac{\partial U}{\partial \hat{E}^{44}} \frac{\hat{g}_{\alpha 4}}{\hat{g}^2_{44}}.$$

2.13 CONSTRUCTING MODELS OF FIELDS

The observations and data on the various interactions lead to new ideas on the specially defined microscopic and macroscopic physical objects called fields. The initial concept of a field was the gravitational field, a concept that grew out of experience. This experience showed that there is an at-a-distance attractive force acting between all material bodies, and also an electromagnetic field, due to the presence and movement of electrically charged or magnetized bodies or to the propagation in vacuo or in neutral media of electromagnetic energy, which may be detected by applying an appropriate test body, a measuring device. (This is the way in which we obtain data on fields in reference frames comoving with the test bodies.)

The theory of fields is a theory of mathematical modeling that enables studying real phenomena. This modeling is based on a number of assumptions about the geometric and physical properties of the interaction mechanisms observed in nature.

As with any modeling process, various models can be built for the same effects. For instance, we know precise ways of interpreting gravitational fields in Newtonian mechanics, on which the basics of the earth's technology and important parts of astronomy rest. But there is another model of gravitational fields, the general theory of relativity, which claims a deeper penetration into the essence of gravity. In this model gravitation is reduced to the geometric properties of the Riemannian four-dimensional space-time, which in turn is determined by processes in matter and in other fields (e.g. electromagnetic) abounding in the Universe or in its parts.

In theoretical physics the models of fields built are connected with clearly idealized assumptions. These models are mathematical constructions that leave out many of the facts firmly established by science. For instance, the macroscopic interpretation of material bodies and fields is related to the hypothesis that bodies and fields are continuous, to the differentiability of the characteristic functions, etc., and especially to the hypothesis that there are fewer degrees of freedom in a small region than the number in the known microscopic structure. The microscopic structure is modeled, in turn, on the basis of the data of refined experiments or theories that are still more refined models reflecting the level of knowledge and understanding of nature in each period of the development of science.

However, for certain classes of phenomena and problems such consciously introduced simplifications and schemes are not only permissible but reasonable and correct when solving many urgent problems of science and technology. Suffice it to note such exactly describable mathematical models as the material points, solid body, elastic body, or the ideal and viscous fluids; also, similar schematic properties are characteristic of any physical objects and processes ever considered in a theory.

Of course, in many cases the problem itself immediately shows whether or not a model is applicable, i.e. whether it reflects the real world sufficiently, is simple, and in general can be used to obtain the needed results, though these will always be only approximate.

Below we will study several methods of building macroscopic models for electromagnetic fields and material media that allow for interaction between the two. In a number of elementary examples of simple models and phenomena, important from the viewpoint of theory, such models exist and are widely used. However, when confronted with complex situations we must design satisfactory, physically justifiable, and universal methods of constructing the models, which is equivalent to introducing macroscopic thermodynamics of mechanics of continuous media and electrodynamics. Such unification of thermodynamics, mechanics, and electrodynamics adds new features to these departments, while the experience gained in the process of building such models enriches their general rational basis, minimizes the number of necessary hypotheses, and unifies their physical and mathematical essence.

In Sec. 2.14 we build a model with due account for the electromagnetic interaction with matter and further elaborate the general methods developed above.

2.14 A MODEL OF INTERACTING MATERIAL MEDIUM AND ELECTROMAGNETIC FIELD

The main characteristic quantities and laws were established and refined over a period of many years starting from the last century. As a result the macroscopic Maxwell equations and their covariant form in the Minkowski space-time become the main basis for electrodynamics. The components $\mathcal{F}_{ij}$ and H_{ij} of antisymmetric four-dimensional second-rank tensors, the components A_i of the vector potential, and the four-dimensional electric current vector I^i $(i, j = 1, 2, 3, 4)$ enter the Maxwell equations as characteristic quantities. These quantities must also be included in the determining parameters of the system encompassing the medium and the field.

We must note here that in the presence of currents, polarization, and magnetization the system of Maxwell equations is not closed and, hence, for a complete description of all concrete and general problems we need additional assumptions.

When we use the basic variational equations, these additional assumptions reduce to establishing the appropriate form of the dependence of the Lagrangian Λ and the functional δW^* on the set of determining parameters. After this is done, by applying the methods developed earlier, we arrive at a closed system consisting of state equations (mechanical and electromagnetic) and the Euler partial differential equations, which must include the Maxwell equations and the universal relationships that reflect the experience of many diversified experiments. As we said before, the selection of Λ and δW^* are connected with a number of universal and particular properties of the model under consideration, namely the field plus the material medium. One such property is the natural requirement that the Euler equations contain all the Maxwell equations.

The main guideline in choosing the Λ and δW^* for the variations δ_d in the basic variational equation must be the assumption that locally the transition to real processes makes the basic variational equation a corollary of the formulation of the given system of the first and second laws of thermodynamics. This choice is implemented on the basis of an analysis of the energy differential equations for a material medium and an electromagnetic field, allowing for the fact that the system including the medium together with the field is not conservative (even in reversible processes of polarization and magnetization), a fact due to the transformation of electromagnetic energy into heat in the presence of currents.

The question of how to establish the expressions for Λ and δW^* via the energy equations in the case of a material field combined with an electromagnetic field is discussed in [19], where the starting relationships for real processes are given by Eqs. (2.5), (2.8), and (2.9) (pp. 262 and 263 in [19]). We recall that in (2.5) and, hence, (2.9) in [19] for the corresponding differentials $dx^i = dx^{*i}$, which are factors of electromagnetic parameters, we have

$$dx^{*i} = u^{*i} d\tau, \quad \text{with} \quad \nabla_k u^{*i} = 0, \tag{2.14.1}$$

where $\mathbf{u}^* = u^{*i} \mathfrak{I}_i^*$ is the velocity 4-vector for the rest inertial frame considered as a system that is comoving with the electromagnetic field and the material medium at a given point in space.

For a material medium,

$$dx^i = u^i d\tau, \quad dx^i = dx^{*i},$$

with $u^i = u^{*i}$, but in general

$$\nabla_k u^i \neq 0 \tag{2.14.2}$$

for a medium. The total energy flux from the electromagnetic field to a unit volume of the medium is given by the formula (see [2], Vol. 1, p. 345)

$$F = -u^{*i} \nabla_k S_i{}^k, \tag{2.14.3}$$

where we have allowed for (2.14.1), and $S_i{}^k$ are the components of the Minkowski tensor:

$$S_i{}^k = [1/(16\pi)]\, \mathcal{F}_{mn} H^{mn} \delta_i^k - [1/(4\pi)]\, \mathcal{F}_{ji} H^{jk}. \tag{2.14.4}$$

In what follows we will use the scalar quantity

$$L = [1/(16\pi)] \mathcal{F}_{mn} H^{mn}. \tag{2.14.5}$$

The energy equation (2.9) in [19] for a system that combines medium and field can be rewritten in the following form:

$$\begin{aligned} &-(dL^* + dU_1^*) + [1/(8\pi)]\, H^{kj} \nabla_i \mathcal{F}_{kj} dx^{*i} + I^k \mathcal{F}_{ki} dx^{*i} \\ &+ dQ^{(e)} + dW_0 - Q_i dx^i = 0. \end{aligned} \tag{2.14.6}$$

Here L^* and U_1^* are quantities calculated for a unit volume dV_{40} conserved in the rest reference frame. The quantity $U_1 = \rho\,(c^2 + U)$, just as in the examples of Secs. 2.11 and 2.12, can be taken as the specific body energy of the medium particles (in the case at hand in the presence of a macroscopic electromagnetic field) calculated per "fluid" volume dV_4, which at a given moment of time is equal to dV_{40}; the quantity $dQ^{(e)}$ is the external influx of heat to a unit volume of field and medium determined from the second law of thermodynamics, and $I^k \mathcal{F}_{ki} \partial x^{*i}$ is the Joule heat per unit volume (this heat is liberated in the course of internal energy processes and is an integral part of the energy flux F over the time $d\tau$; dW_0 is the influx of the external energy of a nonthermal nature across the particle boundary in time $d\tau$; and Q_i is the external four-dimensional generalized body-force density, which determines the influx of the energy $-Q_i dx^i$ to the medium (this quantity has a nonthermal nature).

Let us denote the entropy per unit mass of the medium substance by S and the absolute temperature by T, which we assume exists for processes in the system consisting of field and medium. For real processes the second law of thermodynamics for the above-mentioned system is written thus:

$$\rho T\, dS = dQ^{(e)} + dQ', \tag{2.14.7}$$

where dQ' is the uncompensated heat per unit volume.

Let us assume, for the sake of simplicity, that in the examples at hand the uncompensated heat is produced by only two mechanisms: the liberation of Joule heat and dissipative processes defined by certain tensors with components $\tau_i^{\,k}$ (viscosity, heat conduction, and possibly other mechanisms). This is reflected by the following formula:

$$\begin{aligned} dQ' &= I^k (\nabla_k A_i - \nabla_i A_k)\, u^i\, d\tau + \tau_i^{\,k} \nabla_k (u^i\, d\tau) \\ &= I^k (dx^i \nabla_k A_i - dA_k) + \tau_i^{\,k} \nabla_k\, dx^i. \end{aligned} \tag{2.14.8}$$

The first terms on the right-hand side have a clear physical meaning, since in real phenomena they give the Joule heat, by virtue of the Maxwell equations, which we will derive later from the basic variational equation (2.14.22). The I^k stand for the components of the four-dimensional current, which in an inertial reference frame are

$$I^\alpha = (1/c) j^\alpha, \quad I^4 = -\rho_{(e)}, \tag{2.14.9}$$

where c is the speed of light, j^α the components of the three-dimensional conduction current, and $\rho_{(e)}$ the body charge density.

When irreversible magnetization and polarization is present in the medium, additional terms may appear on the right-hand side of (2.14.8). When we substitute δx^i for dx^i and $\delta A_k = \partial A_k +$

$\delta x^i \nabla_i A_k$, for dA_k, we obtain

$$\delta Q' = I^k (\nabla_k A_i - \nabla_i A_k)\, \delta x^i - I^k \partial A_k + \tau_i{}^k \nabla_k \delta x^i, \tag{2.14.10}$$

while, by virtue of (2.14.8), the expression for the influx of external heat in varied states can be written thus:

$$\delta Q^{(e)} = \rho T \delta S - I^k (\nabla_k A_i - \nabla_i A_k)\, \delta x^i + I^k \partial A_k - \tau_i{}^k\, \nabla_k \delta x^i. \tag{2.14.11}$$

Formulas (2.14.7), (2.14.8), and (2.14.11) can be taken as the definition of the heat increments $dQ^{(e)}$ and $\delta Q^{(e)}$ in terms of the increments of internal determining parameters in real and varied processes.

The quantity $L^* + U_1^*$ stands for the characteristics (defined above) calculated per unit (constant) volume dV_{40} taken in the rest internal reference frame, while by $L + U_1$ we denote similar quantities calculated per unit "fluid" volume dV_4 taken in the reference frame comoving with the medium and at a given moment of time equal to dV_{40}. Since it is clear from the standpoint of physics that

$$(L^* + U_1^*)\, dV_{40} = (L + U_1)\, dV_4, \quad ddV_{40} = 0,$$
$$ddV_4 = \nabla_i\, dx^i dV_4, \tag{2.14.12}$$

we see that

$$-d\,(L^* + U_1^*)\, dV_{40} = -\, d\,[(L + U_1)\, dV_4]$$
$$= -\, [d\,(L + U_1) + (L + U_1)\, \nabla_i dx^i]\, dV_4. \tag{2.14.13}$$

After we substitute $\delta = \partial + \delta x^i \nabla_i$ for d and d^* (and, for one, δx^i for $dx^i = dx^{*i}$) and use formulas (2.14.7), (2.14.11), and (2.14.13), we can write the general energy equations in variations for the system combining field and medium in the following way:

$$\{-\partial\,(L + U_1) + \nabla_k\,[(L + U_1)\, \delta x^k] + \rho T\, \delta S + [1/(8\pi)] H^{kj} \nabla_i \mathscr{F}_{kj} \delta x^i$$
$$+ I^k (\mathscr{F}_{ki} - \nabla_k A_i + \nabla_i A_k)\, \delta x^i + (\nabla_k \tau_i{}^k - Q_i)\, \delta x^i - \nabla_k\, (\tau_i{}^k \delta x^i)$$
$$+ I^k \partial A_k + \delta W_0\}\, dV_4 = 0. \tag{2.14.14}$$

The energy equation in variations, (2.14.14), can be taken as the basis for deriving the basic variational equation in an arbitrary reference frame of the observer, x^i, and with Λ and δW^* that we will now choose.

Depending on the choice of the determining parameters, we can write the functional Λ in similar but slightly different forms (which, however, do not influence its value). For instance, if we assume that L and U_1 are given functions of

$$x^i{}_j,\ S,\ \mathscr{F}_{ij},\ \nabla_k \mathscr{F}_{ij},\ g_{ij},\ K^B\,(\xi^k),\ K^C\,(x^k), \tag{2.14.15}$$

we may, for independent variations δx^i, δS, $\partial .\mathcal{F}_{ij}$, ∂H^{ij}, and ∂A_i and at $\delta K^B = \partial K^C = 0$, put

$$\Lambda = -(L + U_1) + \Lambda',$$
$$\Lambda' = [1/(8\pi)]\mathcal{F}_{ij}H^{ij} - [1/(4\pi)]\nabla_i A_j H^{ij}. \tag{2.14.16}$$

Below we will show that the quantities L and U_1 as functions of the parameters listed in (2.14.15) cannot be given arbitrarily, since from other definitions it follows that $L = (1/16\pi)\, .\mathcal{F}_{ij}H^{ij}$, and later we will see that H^{ij}, in its turn, is expressed in terms of L and U_1 via the state equations.

As parameters μ^A we take the tensor components A_i, $.\mathcal{F}_{ij}$, and H^{ij}. The universal dependence of Λ on A_i and H^{ij} in (2.14.16) is achieved only through Λ', whose concrete form is chosen such that for real processes $\Lambda' \equiv 0$ by virtue of the Maxwell equations (these follow from the basic variation equation as the Euler equations, which is ensured by the presence of Λ' in Λ). When we go over to real processes, the presence of Λ' in no way influences the energy equation (2.14.16).

If we take as independent parameters the following quantities

$$x^i{}_j,\ S,\ M_{ij},\ \nabla_k M_{ij},\ g_{ij},\ K^B(\xi^k),\ K^C(x^k) \tag{2.14.17}$$

with the independent variations being δx^i, δS, ∂M_{ij}, $\partial\mathcal{F}_{ij}$, and ∂A_i and with $\delta K^B = 0$, $\partial K^C = 0$, while

$$4\pi M_{ij} = .\mathcal{F}_{ij} - H_{ij}, \tag{2.14.18}$$

and where we put

$$(1/4)\,.\mathcal{F}_{ij}H^{ij} = L_1, \tag{2.14.19}$$

then we can rewrite the still valid formula (2.14.16) for Λ (after we have excluded H^{ij} with the help of (2.14.18)) in the following form:

$$\Lambda = -(L_1 + U_1) - [1/(16\pi)]\,\mathcal{F}_{ij}.\mathcal{F}^{ij} - [1/(4\pi)]\,.\mathcal{F}^{ij}\nabla_i A_j + M^{ij}\nabla_i A_j \tag{2.14.20}$$

(see [19], p. 274), and assume that L_1 and U_1 are certain functions of the system of parameters (2.14.17) within the concrete physical model.

On the basis of (2.14.14) we find that for δW^* the following formula is valid:

$$\delta W^* = \int\limits_{V_4} \{\rho T\delta S - \mathcal{D}_i\delta x^i + (\nabla_k \tau_i{}^k - Q_i)\delta x^i - \nabla_k(\tau_i{}^k\delta x^i) + I^k\partial A_k\}\, dV_4, \tag{2.14.21}$$

where

$$\mathcal{D}_i = -[1/(8\pi)]\, H^{kj}\nabla_i\mathcal{F}_{kj} - I^k(\mathcal{F}_{ki} - \nabla_k A_i + \nabla_i A_k).$$

As to δW, this is a functional found from the basic variational equation

$$\delta \int\limits_{V_4} \Lambda dV_4 + \delta W^* + \delta W = 0, \tag{2.14.22}$$

which in terms of the arguments for $L + U_1$ given in (2.14.15) has the following form:

$$\begin{aligned}\delta W &= \int\limits_{\Sigma} (P'{}_i{}^k \delta x^i + N^{sk} \partial A_s + M^{pqk} \partial \mathscr{F}_{pq})\, n_k\, d\sigma \\ &= \int\limits_{\Sigma} (P_i{}^k \delta x^i + N^{sk} \delta A_s + M^{pqk} \delta \mathscr{F}_{pq})\, n_k\, d\sigma.\end{aligned} \tag{2.14.23}$$

If we now use the general formulas discussed in Sec. 2.9 and employ additional parameters μ^A, which are A_i, H^{ij}, and $\mathscr{F}_{ij}$, we obtain the following Euler equations. At the variations ∂H^{ij} and ∂A_i there are the Maxwell equations

$$\mathscr{F}_{ij} = \nabla_i A_j - \nabla_j A_i, \tag{2.14.24}$$

$$\nabla_i H^{ki} = 4\pi I^k. \tag{2.14.25}$$

The formula for the temperature stands at the variation δS:

$$\rho T = \partial (L + U_1)/\partial S \tag{2.14.26}$$

Finally, at the variation $\partial \mathscr{F}_{ij}$ there stands the state equation for polarization and magnetization

$$[1/(8\pi)] H^{ij} = \partial (L + U_1)/\partial \mathscr{F}_{ij} - \nabla_k \partial (L + U_1)/\partial \nabla_k \mathscr{F}_{ij}, \tag{2.14.27}$$

where we have allowed for the fact that $\mathscr{F}_{ij}$ and H^{ij} are antisymmetric in their indices. Obviously, when $L + U_1$ does not depend on $\nabla_k \mathscr{F}_{ij}$, formula (2.14.27) simplifies considerably.

On the basis of the general relationship (2.14.23) for δW, we find from (2.14.22) that

$$P'{}_i{}^k = [\partial (L + U_1)/\partial x^i{}_j]\, x^k{}_j + (L + U_1)\, \delta_i{}^k + \tau_i{}^k, \tag{2.14.28}$$

$$N^{sk} = -[1/(4\pi)] H^{sk} \tag{2.14.29}$$

and

$$M^{pqk} = \partial (L + U_1)/\partial \nabla_k \mathscr{F}_{pq}. \tag{2.14.30}$$

If we allow for (2.14.29) and (2.14.30), for $P_i{}^k$ we obtain:

$$P_i{}^k = P'{}_i{}^k + [1/(4\pi)] H^{sk} \nabla_i A_s - [\partial (L + U_1)/\partial \nabla_k \mathscr{F}_{pq}] \nabla_i \mathscr{F}_{pq}. \tag{2.14.31}$$

In the case at hand, from the formula (2.14.21) for δW^* and the Euler equations, which contain the Maxwell equations, we have

$$\mathscr{D}_i = -[1/(8\pi)]\, H^{pq} \nabla_i \mathscr{F}_{pq},$$

while the term with $I^k \partial A_k$ yields

$$M_A = I^k = [1/(4\pi)] \nabla_s H^{ks}.$$

For this reason Eq. (2.9.7) takes the form

$$\nabla_k P_i{}^k = Q_i + [\partial(L+U_1)/\partial K^C] \nabla_i K^C - [1/(8\pi)] H^{pq} \nabla_i \mathscr{F}_{pq}$$
$$+ [1/(4\pi)] (\nabla_s H^{ks}) \nabla_i A_k. \qquad (2.14.32)$$

In this equation we transform the last two terms thus:

$$-\frac{1}{8\pi} H^{pq} \nabla_i \mathscr{F}_{pq} + \frac{1}{4\pi} (\nabla_s H^{ks}) \nabla_i A_k = -\frac{1}{8\pi} H^{pq} \nabla_i (\nabla_p A_q - \nabla_q A_p)$$
$$+ \frac{1}{4\pi} \nabla_s (H^{ks} \nabla_i A_k) - \frac{1}{4\pi} H^{ks} \nabla_i \nabla_s A_k = \frac{1}{4\pi} \nabla_k (H^{sk} \nabla_i A_s). \qquad (2.14.33)$$

Since the sum of the first and third terms in the intermediate expression is zero, the only term remaining is the second, where in the final formula the summation indices k and s are interchanged.

Obviously, if we substitute the formula (2.14.31) for $P_i{}^k$ in Eq. (2.14.32), then on the right- and left-hand sides of the resulting equation the last terms, which are caused by the presence of an electromagnetic field, are not gauge invariant but by virtue of (2.14.31) cancel out. Gauge invariance is preserved also at a strong discontinuities $[P_i{}^k n_k] = 0$ if we take into account that $[H^{ik} n_k] = 0$.

If in addition to the tensor $\mathbf{P} = P_i{}^k \mathbf{Э}^i \mathbf{Э}_k$ we introduce the tensor $\mathbf{T} = T_i{}^k \mathbf{Э}^i \mathbf{Э}_k$ (see Sec. 2.9) according to the formula

$$T_i{}^k = x^k{}_j \, \partial(L+U_1)/\partial x^i{}_j + (L+U_1)\, \delta^k_i + \tau_i{}^k - [\partial(L+U_1)\, \partial \nabla_k \mathscr{F}_{pq}] \nabla_i \mathscr{F}_{pq}$$
$$= P^{*k}_i + x^k_j \, \partial L/\partial x^i_j + L \delta^j_i = P_i{}^k - [1/(4\pi)]\, H^{sk} \nabla_i A_s$$
$$= P'_i{}^k - [\partial(L+U_1)/\partial \nabla_k \mathscr{F}_{pq}] \nabla_i \mathscr{F}_{pq},$$

we can rewrite Eq. (2.14.32) in the following form:

$$\nabla_k T_i{}^k = Q_i + [\partial(L + U_1)\, \partial K^C] \nabla_i K^C. \qquad (2.14.34)$$

The right-hand side of this equation is zero when

$$Q_i = 0, \quad [\partial(L + U_1)/\partial K^C] \nabla_i K^C = 0.$$

Thus, in modified form the momentum equation (2.14.34) has a divergence form, and according to the definition commonly used in the literature the tensor $\mathbf{T} = T_i{}^k \mathbf{Э}^i \mathbf{Э}_k$ is called the energy-momentum tensor of the medium and electromagnetic field. This tensor was introduced in [8] and it can be used instead of $\mathbf{P}$. In some cases, where there is no field, $\mathbf{T} = \mathbf{P}$.

The system of partial differential equations (2.14.24)-(2.14.32), first constructed in [8], constitutes a complete system of equations when I^k, $\tau_i{}^k$, and Q_i are given and when L and U_1 are presented as functions of the parameters (2.14.15). But these two functions must be related by the following differential law:

$$2L = \mathcal{F}_{ij}\partial(L+U_1)/\partial\mathcal{F}_{ij} - \mathcal{F}_{ij}\nabla_k\partial(L+U_1)/\partial\nabla_k\mathcal{F}_{ij}, \tag{2.14.35}$$

which must be satisfied identically and which follows from (2.14.28) after multiplication into $\mathcal{F}_{ij}$ and contraction with respect to the indices i and j.

2.15 EXAMPLES

We will now turn to examples for which the following assumptions are true:

$$\begin{aligned} L &= [1/(16\pi)]H^{ij}\mathcal{F}_{ij} = [1/(16\pi)]c^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl} \text{ and} \\ U_1 &= \rho(c^2 + U), \end{aligned} \tag{2.15.1}$$

where the tensor components c^{ijkl} and the scalar U depend solely on

$$x^i_j,\ S,\ g_{ij},\ K^C. \tag{2.15.2}$$

In (2.15.1) the quantities c^{ijkl}, $\mathcal{F}_{ij}$, and $\mathcal{F}_{kl}$ are taken in any same system of coordinates of the observer, which may coincide with the comoving system. Obviously, the following symmetry conditions are assumed valid

$$c^{ijkl} = c^{klij} = -c^{jikl} = -c^{ijlk}.$$

In view of (2.15.1) and (2.15.2), the condition (2.14.35) is satisfied identically, and the components $P'_i{}^k$ expressed in terms of U_1 are determined via (2.11.12) if U is determined via (2.11.9).

The relation between the electromagnetic field and the phenomena in the substance (the medium) is achieved only through the dependence of the c^{ijkl} on $x^i{}_j$, S, g_{ij}, and K^C. In this case the state equation (2.14.27) and the formulas for the energy-momentum tensor component $T_a{}^c$ can be written thus:

$$H^{ij} = c^{ijkl}\mathcal{F}_{kl}, \text{ with } c^{ijkl} = c^{ijkl}(x^i{}_j,\ S,\ g_{ij},\ K^C) \tag{2.15.3}$$

and

$$T_a{}^c = P^*{}_a{}^c + [1/(16\pi)]\,x^c{}_j\,(\partial c^{pqmn}/\partial x^a{}_j)\,\mathcal{F}_{pq}\mathcal{F}_{mn} + L\delta^c_a, \tag{2.15.4}$$

where $P^*{}_a{}^c$ is the part of $T_a{}^c$ independent of L. According to (2.15.1) we can write

$$L = [1/(16\pi)]\,\hat{c}^{ijkl}\hat{\mathcal{F}}_{ij}\hat{\mathcal{F}}_{kl} = [1/(16\pi)]\,\tilde{c}^{pqrs}\tilde{\mathcal{F}}_{pq}\tilde{\mathcal{F}}_{rs},$$

where

$$\tilde{c}^{pqrs} = \hat{c}^{ijkl}A^p{}_iA^q{}_jA^r{}_kA^s{}_l,\quad \tilde{\mathcal{F}}_{ab} = B^i{}_aB^j{}_b\hat{\mathcal{F}}_{ij}, \tag{2.15.5}$$

with the components $\hat{c}^{ijkl}$ taken in the comoving system of coordinates, and $A^n{}_m$ and $B^d{}_c$ elements of mutually inverse matrices.

The transformations (2.15.5) can be considered as transformations at a given point of the tensor components taken in the basis tetrad $\hat{\text{Э}}_i$ of the comoving coordinate system into another basis tetrad $\tilde{\text{Э}}_i$. A transformation of this type is given by a matrix whose elements are $A_m{}^n$; these matrix elements may depend on any variables, given or sought, and, for one, where the tetrad $\tilde{\text{Э}}_i$ coincides with the observer's tetrad Э_i, i.e.

$B^d{}_c = \partial x^d/\partial \xi^c = x^d{}_c$ and $A^m{}_n = \partial \varepsilon^m/\partial x^n = \xi_n{}^m\,(x^d{}_c)$.

Obviously, when we have to calculate the derivatives $\partial L/\partial x^i{}_j$, with the components $\tilde{c}^{ijkl}$ and $\tilde{\mathcal{F}}_{ij}$ expressed in different bases, the result cannot depend on the properties of the transformations (2.15.5) with coefficients $A^n{}_m$ and $B^d{}_c$, since the scalar L does not contain these coefficients.

If we allow for the fact that, by definition, the variations of tensors are tensors of the same nature (see (2.4.5) and (2.4.6)), then we have

$$\partial\hat{\mathcal{F}}_{ab} = \partial\mathcal{F}_{ij}x^i{}_a x^j{}_b,$$

whereby after the variations are realized and the coefficients of δx^i and $\partial\mathcal{F}_{ij}$ or of δx^i and $\partial\hat{\mathcal{F}}_{pq}$ are nullified, we arrive at the same Euler equations at $|\,x^i{}_j\,| \neq 0$.

It follows from the aforesaid that it is expedient to consider the components $\mathcal{F}_{ab}$ and c^{pqmn} in (2.15.4) in the comoving system of coordinates; the same is true of the components $\hat{c}^{pqmn}$ as functions of the determining parameters (also taken in comoving systems), and after we have applied (2.15.4) to these parameters (the arguments of the above-mentioned components) we must substitute for them the components of the same tensors but taken in the observer's coordinate system, in accordance with the general theory developed in Sec. 2.9. This trick is useful since in a comoving system of coordinates $x^i{}_j = \delta^i_j$. For this reason the number of important tensor arguments (the components with "hats") diminishes.

If we assume that the $\hat{c}^{pqmn}$ depend not only on various invariant quantities but also on the components $x^i{}_j$ and g^{ij} and certain other tensor components of the K^B type (not depending on the sought functions $x^i{}_j\,(\xi^k)$), in a comoving reference frame the arguments $x^i{}_j = \delta^i_j$ become fixed (known) parameters of the K^B type. For this reason, in comoving coordinate systems, with the determining parameters also taken in a comoving system, instead of the parameters $x^i{}_j$ and $\hat{g}^{ij}$ we have the following set of parameters: $\hat{u}^i$, $\hat{g}^{ij}$, $\hat{K}^B$, where $\hat{u}^i$ are the components of a unit vector $\mathbf{u}$ in a comoving system of coordinates and directed along the tangents to the world lines of points in the medium, with $\hat{u}^i = \hat{\overset{*}{u}}{}^i$, where $\hat{\overset{*}{u}}{}^i$ are the components of the unit vector $\mathbf{u}^*$ in a local proper tetrad.

In the general case, in the system $\hat{\mathrm{Э}}_i$ we have

$d\mathbf{u}^*/ds = 0, \quad d\mathbf{u}/ds = \mathbf{a},$

where $\mathbf{a}$ is the absolute acceleration of a point in the moving medium in the comoving system of coordinates, and $ds = (\hat{g}_{44})^{1/2}\, d\xi^4$ is the arc element of a world line.

When introducing the vector arguments of type $\mathbf{u}$ it is important to note the following. In general, when the observer's reference frame and the comoving reference frame are different, $x^i{}_j \neq \delta^i_j$. Then at each point in the medium two different unit vectors can be introduced: $\mathbf{u}_x$ and $\mathbf{u}_\xi$, which are directed along the world lines x^4 and ξ^4 of the observer's and the comoving reference frame. By definition we have

$$\mathbf{u}_x = \frac{\partial x^k}{\partial x^4} \frac{1}{\sqrt{g_{44}(x^k)}} \mathrm{Э}_k \quad \text{and}$$
$$\mathbf{u}_\xi = \frac{\partial \xi^i}{\partial \xi^4} \frac{1}{\sqrt{\hat{g}_{44}(\xi^k)}} \mathrm{Э}_i. \tag{2.15.6}$$

Since $\mathrm{Э}_k = (\partial \xi^i/\partial x^k)\, \hat{\mathrm{Э}}_i$, it follows from (2.15.6) that

$$\hat{u}^i_x = \delta^k_4 \frac{1}{\sqrt{g_{44}(x^s)}} \frac{\partial \xi^i}{\partial x^k} = \delta^k_4 \frac{1}{\sqrt{g_{44}(x^s)}} \xi_k{}^i, \tag{2.15.7}$$

$$\hat{u}^i_\xi = \frac{\partial x^k}{\partial \xi^4} \frac{1}{\sqrt{\hat{g}_{44}(\xi^s)}} \frac{\partial \xi^i}{\partial x^k} = x^k{}_4 \frac{1}{\sqrt{g_{pq} x^p{}_4 x^q{}_4}} \xi_k{}^i. \tag{2.15.8}$$

Obviously, if the observer's system of coordinates coincides at any given point with the comoving system or with a local rest system, we have

$$\hat{u}^i_x = \hat{u}^i_\xi = \hat{u}^i = \left(1/\sqrt{g_{44}(x^s)}\right) \delta^k_4 \xi_k{}^i = u^k \xi_k{}^i (x^d{}_c), \tag{2.15.9}$$

with $u^k = u^k{}_x$ in this case.

On the basis of (2.15.9) in the case under discussion we see that the components $\hat{c}^{pqrs}$ for any comoving coordinate system can be considered to be tensor functions of the invariants and the components $\hat{g}^{ij}$, $\hat{u}^i$, and $\hat{K}^B$. When anisotropy is present, we will have tensor components of the $\hat{K}^B$ type; when these components are zero, then by definition the material medium is isotropic in its electromagnetic properties. If the second is the case, the theory of nonlinear tensor functions (see [2]) implies that

$$\hat{c}^{ijkl}(\hat{g}^{ij}, \hat{u}^i) = (1/2)\,[(1/\mu)\,(\hat{g}^{ik}\hat{g}^{jl} - \hat{g}^{il}\hat{g}^{jk})$$
$$+ (\varepsilon - 1/\mu)\,(\hat{g}^{ik}\hat{u}^j\hat{u}^l - \hat{g}^{jk}\hat{u}^i\hat{u}^l + \hat{g}^{jl}\hat{u}^i\hat{u}^k - \hat{g}^{il}\hat{u}^j\hat{u}^k)]. \tag{2.15.10}$$

Here ε and μ are scalars (the permittivity and permeability, respectively) that may be functions of ρ and S or of ρ and T, and also of other invariant variables (e.g. invariants related to $\hat{E}^{pq}$ and $\hat{g}_{rs}$), etc.

In the orthonormal tetrad $\hat{\partial}_i$ we have $\hat{g}^{ij} = 0$ at $i \neq j$ and $\hat{g}^{11} = \hat{g}^{22} = \hat{g}^{33} = -1$, $\hat{g}^{44} = 1$, and $\hat{u}^4 = 1$ and $\hat{u}^\alpha = 0$. This implies that

$$\hat{c}^{\alpha\beta\alpha\beta} = 1/(2\mu) \quad \text{at} \quad \alpha \neq \beta,$$

$$\hat{c}^{\alpha 4 \alpha 4} = \hat{c}^{4\alpha 4\alpha} = (1/2)\,[-1/\mu - (\varepsilon - 1/\mu)] = -\varepsilon/2,$$

while the other components are zero.

Since

$$\hat{H}^{pq} = \hat{c}^{pqrs} \hat{\mathcal{F}}_{rs},$$

we find that

$$\hat{H}^{\alpha\beta} = (1/\mu)\,\hat{\mathcal{F}}_{\alpha\beta} \quad \text{and} \quad \hat{H}^{\alpha 4} = -\varepsilon \hat{\mathcal{F}}_{\alpha 4}.$$

From this it follows that

$$\mathbf{B} = \mu \mathbf{H} \quad \text{and} \quad \mathbf{D} = \varepsilon \mathbf{E}, \tag{2.15.11}$$

where **B**, **D**, **H**, E are known vectors defined in the proper coordinate system.

In the case of anisotropy, the formulas (2.15.10) and (2.15.11) become more complex. The formulas (2.15.11), written componentwise, must be modified so that μ^{-1} and ε are three-dimensional spatial tensors whose form is determined by tensors that define the anisotropy (see [2, 3]). When we go over from the arguments for $\hat{c}^{ijkl}$ in the comoving frame to the arguments in the observer's reference frame, which do not generally coincide, we must bear in mind that $\hat{g}^{ij} = g^{mn}\xi_m{}^i\xi_n{}^j$ and that the components $\hat{u}^i$ can be substituted by the $\hat{u}^i_x$ by (2.15.7) or by $\hat{u}^i_\xi$ by (2.15.8), or one of the vectors with components $\hat{u}^i$ can be substituted by $\hat{u}^i_x$ while the other can be substituted by $\hat{u}^i_\xi$ (these vectors coincide only when the observer's reference frame coincides with the proper or comoving frame); other examples can also be cited.

In accordance with what has just been said, the same formula (2.15.10) leads to two basic variants for the components $\hat{u}^i$ in the comoving system of coordinates.[9]

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[9] Thus, since the problem of fixing the energy-momentum tensor of an electromagnetic field cannot be solved in a unique manner, the $\hat{c}^{pqrs}$ has no unique solution either. Variant 1 is studied in [8].

Variant 1: $\hat{u}^i \to u^k_x \xi_k{}^i$. In this case according to (2.15.7) we have

$$\hat{c}^{pqrs} = \frac{1}{2} \xi_i{}^p \xi_j{}^q \xi_k{}^r \xi_l{}^s \left[\frac{1}{\mu} (g^{ik} g^{jl} - g^{il} g^{jk}) + \left(\varepsilon - \frac{1}{\mu} \right) \frac{1}{g_{44}(x^s)} \right.$$

$$\left. \times (g^{ik} \delta^j_4 \delta^l_4 - g^{jk} \delta^i_4 \delta^l_4 + g^{jl} \delta^i_4 \delta^k_4 - g^{il} \delta^k_4 \delta^j_4) \right]. \tag{2.15.12}$$

Variant 2: $\hat{u}^i \to u^k_\xi \xi_k{}^i$. In this case on the basis of (2.15.8) we can write

$$\hat{c}^{pqrs} = \frac{1}{2} \xi_i{}^p \xi_j{}^q \xi_k{}^r \xi_l{}^s \left[\frac{1}{\mu} (g^{ik} g^{jl} - g^{il} g^{jk}) \right.$$

$$\left. + \left(\varepsilon - \frac{1}{\mu} \right) \frac{1}{g_{mn} x^m{}_4 x^n{}_4} (g^{ik} x^j{}_4 x^l{}_4 - g^{jk} x^i{}_4 x^l{}_4 + g^{jl} x^i{}_4 x^k{}_4 - g^{il} x^k{}_4 x^j{}_4) \right]. \tag{2.15.13}$$

It is obvious that the structure of the formulas (2.15.12) and (2.15.13) established for the particular case of the argument components of the determining parameters remains the same in more complex cases, when there are additional tensor arguments $\hat{K}^C$ and $\hat{\mu}^A$, which are tensor components in the comoving system of coordinates, or components of the same tensors K^C and μ^A in the observer's system.

In the general case we have the following formula:

$$\hat{c}^{pqrs} (\hat{g}^{ij}, x^i{}_j, \hat{K}^C, \hat{\mu}^A) = \xi_i{}^p \xi_j{}^q \xi_k{}^r \xi_l{}^s \hat{c}^{ijkl} (g^{ij}, x^i{}_j, K^C, \mu^A). \tag{2.15.14}$$

We can rewrite formula (2.15.3) thus:

$$\hat{H}^{pq} = \hat{c}^{pqrs} \hat{\mathscr{F}}_{rs} = \xi_i{}^p \xi_j{}^q \xi_k{}^r \xi_l{}^s \hat{c}^{ijkl} (g^{ij}, x^i{}_j, K^C, \mu^A) \hat{\mathscr{F}}_{rs}. \tag{2.15.15}$$

We will now introduce a general formula that contains only derivatives of the $\hat{c}^{ijkl} (g^{ij}, x^i{}_j, K^C, u^A)$ with respect to $x^a{}_b$. We have

$$x^c{}_b \frac{\partial \hat{c}^{pqrs}}{\partial x^a{}_b} \hat{\mathscr{F}}_{pq} \hat{\mathscr{F}}_{rs}$$

$$= x^c{}_b \frac{\partial \xi_i{}^p \xi_j{}^q \xi_k{}^r \xi_l{}^s \hat{c}^{ijkl} (g^{ij} \ldots)}{\partial x^a{}_b} \hat{\mathscr{F}}_{pq} \hat{\mathscr{F}}_{rs}. \tag{2.15.16}$$

By virtue of (2.12.9) we can write

$$\partial \xi_i{}^p / \partial x^a{}_b = -\xi_i{}^b \xi_a{}^p$$

and similar formulas for the other derivatives. When we have done all the differentiations, used the symmetry properties of the $\hat{c}^{ijkl}$ and $\hat{\mathscr{F}}_{mn}$, rearranged the indices, and used (2.15.15), we arrive at the following result:

$$\frac{1}{16\pi} x^{c}{}_{b} \frac{\partial \hat{c}^{pqrs}}{\partial x^{a}{}_{b}} \hat{\mathscr{F}}_{pq} \hat{\mathscr{F}}_{rs} = -\frac{1}{4\pi} x^{c}{}_{m} \hat{H}^{mq} \xi_{a}{}^{p} \hat{\mathscr{F}}_{pq}$$

$$+\frac{1}{16\pi} x^{c}{}_{b} \frac{\partial \hat{c}^{ijkl}}{\partial x^{a}{}_{b}} \mathscr{F}_{ij} \mathscr{F}_{kl}. \tag{2.15.17}$$

We then substitute (2.15.17) into (2.15.4) and, bearing in mind that $x^{c}{}_{m} \hat{H}^{mq} \xi_{a}{}^{p} \hat{\mathscr{F}}_{pq} = H^{cs} \mathscr{F}_{as}$, obtain

$$T_{a}{}^{c} = x^{c}{}_{j}\, \partial U_{1} / \partial x^{a}{}_{j} + U_{1} \delta_{a}^{c} + \tau_{a}{}^{c} + S_{a}{}^{c} + \tilde{T}_{a}{}^{c}, \tag{2.15.18}$$

where $S_{a}{}^{c}$ are the components of the Minkowski tensor:

$$S_{a}{}^{c} = -[1/(4\pi)]\,[\mathscr{F}_{sa} H^{sc} - (1/4)\, \mathscr{F}_{mn} H^{mn} \delta_{a}^{c}];$$

we also know that

$$\tilde{T}_{a}{}^{c} = [1/(16\pi)]\, x^{c}{}_{a} (\partial \hat{c}^{ijkl} / \partial x^{a}{}_{b})\, \mathscr{F}_{ij} \mathscr{F}_{kl}.$$

The last term in (2.15.18), which we denoted by $\tilde{T}_{a}{}^{c}$, yields the components of the magnetostriction and electrostrictional parts of the energy-momentum tensor.

Obviously, in variant 1 we obtain, by virtue of (2.15.12), the following result:

$$\tilde{T}_{a}{}^{c} = 0 \tag{2.15.19}$$

if μ and ε are constants.

But if μ and ε are functions of the entropy S and the density ρ or the temperature T and ρ, then in variant 2 we obtain, by virtue of (2.15.13),

$$\tilde{T}_{a}{}^{c} = \frac{1}{4\pi} \left(\varepsilon - \frac{1}{\mu} \right) u^{c} u^{n} \mathscr{F}_{mn} \mathscr{F}^{m}{}_{q} (\delta_{a}^{q} - u^{q} u_{a})$$

$$+ \frac{1}{8\pi} \left[\frac{\rho}{\mu^{2}} \frac{\partial \mu}{\partial \rho} \mathscr{F}_{mn} \mathscr{F}^{mn} - \rho \left(\frac{1}{\mu^{2}} \frac{\partial \mu}{\partial \rho} + \frac{\partial \varepsilon}{\partial \rho} \right) \right.$$

$$\left. \times \mathscr{F}_{mn} \mathscr{F}^{n}{}_{s} u^{m} u^{s} \right] (\delta_{a}^{c} - u_{a} u^{c}). \tag{2.15.20}$$

Here $u^{l} = dx^{l}/ds$ and $u_{i} = g_{il} u^{l}$.

In the first term in (2.15.20) and in a Cartesian proper system of coordinates only the matrix elements (representing the corresponding components of the tensor) that stand in the first three rows of the fourth column are nonzero. In variant 2 considered here, the full form for $T_a{}^c$ is

$$T_a{}^c = x^c{}_j \frac{\partial U_1}{\partial x^a{}_j} + U_i \delta^c_a + \tau_a{}^c + A_a{}^c + \frac{1}{8\pi}\left[\frac{\rho}{\mu}\frac{\partial \mu}{\partial \rho}\mathcal{F}_{mn}\mathcal{F}^{mn} - \rho\left(\frac{1}{\mu^2}\frac{\partial \mu}{\partial \rho} + \frac{\partial \varepsilon}{\partial \rho}\right) \times u^m u^s \mathcal{F}_{mn}\mathcal{F}^n{}_s\right](\delta^c_a - u_a u^c). \qquad (2.15.21)$$

Here

$$A_a{}^c = S_a{}^c + [1/(4\pi)]\,(\varepsilon - 1/\mu)\, u^c u^n \mathcal{F}_{mn}\mathcal{F}^m{}_q\,(\delta^q_a - u^q u_a) \qquad (2.15.22)$$

are the components of the Abraham tensor.

For an electromagnetic field in vacuo or in a neutral nonpolarizable and nonmagnetizable medium ($\mu = 1$ and $\varepsilon = 1$), the state equation (2.15.3) takes the form

$$H^{ij} = g^{ip}g^{jq}\mathcal{F}_{pq}, \quad \text{or} \quad H_{pq} = \mathcal{F}_{pq},$$

and on the basis of (2.15.18) and (2.15.21) we obtain

$$T_a{}^c = x^c{}_j \partial U_1/\partial x^a{}_j + U_1\delta^c_a + \tau_a{}^c + S_a{}^c.$$

In this case the Minkowski tensor proves to be the symmetric Maxwell tensor, with $\nabla_c S_a{}^c = (1/c)\mathcal{F}_{ma}I^m$.

If electric currents are not present, all the Euler equations split into two closed systems, one for the medium and one for the field.

Several details of this article have been discussed at length by some of my pupils; most of whom are the authors of [11-18]. I wish to thank all of them.

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3

Transition from Relativistic to Newtonian Mechanics in the Presence of Irreversible Processes

L. T. Chernyi, Cand.Sc. (Physics and Mathematics)
Institute of Mechanics, M. V. Lomonosov Moscow State University

3.1 THE BASIC VARIATIONAL EQUATION

According to the special relativity theory, three-dimensional physical space and one-dimensional time form the four-dimensional Minkowski space-time. Suppose that x^k is the observer's system of coordinates in the Minkowski space-time, and $g_{ij}(x^k)$ are the metric tensor components in this system.[1] In the x^k system of coordinates corresponding to a synchronous inertial reference frame[2] connected with the observer we have

$$x^4 = ct, \quad g_{ij} = \left\| \begin{array}{c:c} -h_{\alpha\beta} & 0 \\ \hdashline 0 & 1 \end{array} \right\|, \tag{3.1.1}$$

where c is the speed of light in vacuum, t the time measured in the observer's reference frame, and $h_{\alpha\beta}$ are quantities that generally depend on the spatial coordinate x^γ but in special relativity can always be chosen so that in the entire space $h_{\alpha\beta} = 1$ at $\alpha = \beta$ and $h_{\alpha\beta} = 0$ at $\alpha \neq \beta$. (In general relativity this can be achieved locally or along any given curve. The corresponding systems of coordinates are known as the Fermi normal coordinates.)

If we wish to describe the motion of a continuous medium, then besides the observer's system of coordinates x^k we must introduce a coordinate system ξ^j comoving with the medium. These coordinates ($j = 1, 2, 3$) individualize infinitely small particles of the medium and are called the Lagrangian coordinates. The quantity ξ^4 is a parameter of points on the world lines of the continuous medium with $\xi^\alpha = \text{const}$. The relationship between the x^k and ξ^j coordinates is established by the law of motion of the medium

$$x^k = x^k(\xi^j). \tag{3.1.2}$$

The four-dimensional velocity of the medium $\mathbf{u}$ is a unit vector tangent to the world lines of the medium. In the observer's system of coordinates it has the components

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[1] It is understood that Latin indices take on values 1, 2, 3, or 4, while Greek indices take on values 1, 2, or 3.

[2] We assume that all inertial reference frames are synchronous.

$$\begin{aligned} u^k &= \left(\frac{dx^k}{ds}\right)_{\xi\gamma} = \frac{dx^k}{c d\tau} = \left\| \frac{v^\alpha}{\sqrt{c^2-v^2}} \ \vdots\ \frac{c}{\sqrt{c^2-v^2}} \right\|, \\ u_k &= g_{ki}u^i = \left\| \frac{-v_\alpha}{\sqrt{c^2-v^2}} \ \vdots\ \frac{c}{\sqrt{c^2-v^2}} \right\|, \end{aligned} \tag{3.1.3}$$

where

$$\begin{aligned} &ds = \sqrt{g_{ij}(dx^i)_{\xi\gamma}(dx^j)_{\xi\gamma}} = \sqrt{c^2-v^2}\,(dt)_{\xi\gamma} \equiv c\,d\tau, \\ &v^\alpha = \left(\frac{dx^\alpha}{dt}\right)_{\xi\gamma}, \quad v^2 = h_{\alpha\beta}v^\alpha v^\beta, \quad v_\alpha = h_{\alpha\beta}v^\beta, \end{aligned} \tag{3.1.4}$$

ds is the arc element of a world line of the medium, $d\tau$ is the proper time increment, and v^α are the components of three-dimensional velocity in the observer's inertial reference frame.

The constructing of models of continuous media always involves some postulates. The most rational way to do this is to start at the basic variational equation [1-3]

$$\delta (1/c) \int\limits_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0, \tag{3.1.5}$$

where[3]

$$dV_4 = \sqrt{-g}\, d^4x = \sqrt{-\hat{g}}\, d^4\xi = d\hat{V}_3\, ds, \quad -\Lambda\, d\hat{V}_3 = d\mathscr{E}; \tag{3.1.6}$$

$$g = |g_{ij}|, \ \hat{g} = |\hat{g}_{ij}|, \ d^4x = dx^1 \ldots dx^4, \ d^4\xi = d\xi^1 \ldots d\xi^4, \tag{3.1.7}$$

Λ is the Lagrangian of the continuous medium, δW^* a fixed functional that accounts for irreversible processes and external actions, δW a functional determined from Eq. (3.1.5) and represented by an integral over the surface enveloping the region V_4 in the Minkowski space-time and, possibly, over both sides of strong discontinuities inside V_4, $d\hat{V}_3$ is the three-dimensional volume of an individual infinitely small particle of the continuous medium in the comoving system of coordinates, and $d\mathscr{E}$ is the internal energy of the same particle. The variations in Eq. (3.1.3) are taken at ξ^j = const.

Therefore, a specific model of a continuous medium in this setting is fixed by choosing the Lagrangian and the functional δW^*. For instance, if we put

$$\Lambda = -\rho c^2 - \rho U(S, \hat{\gamma}_{ij}, \hat{\mathring{\gamma}}_{ij}, \hat{K}_B), \tag{3.1.8}$$

$$\delta W^* = (1/c) \int\limits_{V_4} (\rho T \delta S - Q_i \delta x^i)\, dV_4, \quad Q_i u^i = (1/c)(dQ^e/d\tau), \tag{3.1.9}$$

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[3] In what follows, the quantities referring to the comoving system of coordinates will be labeled by a "hat" over the symbol.

where

$$\hat{\gamma}_{ij} = \hat{u}_i \hat{u}_j - \hat{g}_{ij}, \quad \hat{u}_i = \hat{g}_{i4} \Big/ \sqrt{\hat{g}_{44}} = \hat{g}_{ij}\, d\xi^j/ds, \tag{3.1.10}$$

$$\hat{g}_{ij} = g_{kr} x_i^k x_j^r, \quad x_i^k \equiv \partial x^k / \partial \xi^i, \tag{3.1.11}$$

$$\rho = \mathring{\rho} \sqrt{\mathring{\gamma}/\hat{\gamma}}, \quad \hat{\gamma} = |\hat{\gamma}_{\alpha\beta}|, \quad \mathring{\gamma} = |\hat{\mathring{\gamma}}_{\alpha\beta}|, \tag{3.1.12}$$

and all the parameters $\hat{K}_B$ are known functions of Lagrangian coordinates ξ^α (these functions remain unvaried), we have the model of a perfectly elastic body in which all processes are reversible by definition. The arguments of the Lagrangian and the functional δW^* have the following physical meaning: ρ the rest mass density of the medium, S the specific entropy of the medium, T the absolute temperature, Q_i the components of the 4-vector force density, $dQ^{(e)}$ the body external influx of heat in time $d\tau = ds/c$ (for more complex models of media, $Q_i u^i$ may also include influxes of energy other than thermal), $\hat{g}_{ij}$ and $\hat{u}_i$ the components of the metric tensor and the velocity 4-vector of the medium in the comoving coordinate system, $\hat{\gamma}_{ij}$ the components of the spatial metric tensor satisfying the relationships [4]

$$u^i \gamma_{ij} = u^j \gamma_{ij} = 0, \quad \gamma^*_{4j} = \gamma^*_{i4} = 0, \quad \hat{\gamma}_{4j} = \hat{\gamma}_{i4} = 0 \tag{3.1.13}$$

and defining, in a comoving coordinate system, the length dl of infinitesimal individualized segments in the continuous medium that correspond to infinitesimal increments $d\xi^\alpha$ of Lagrangian coordinates,

$$dl^2 = \hat{\gamma}_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta = \hat{\gamma}_{ij}\, d\xi^i\, d\xi^j, \tag{3.1.14}$$

$\mathring{\rho}$ is the medium's rest mass density in an "initial" state and $\mathring{\gamma}_{ij}$ are the components of the spatial metric tensor in the initial state, which satisfy the relationships

$$u^i \mathring{\gamma}_{ij} = u^j \mathring{\gamma}_{ij} = 0, \quad \mathring{\gamma}^*_{4j} = \mathring{\gamma}^*_{i4} = 0, \quad \hat{\mathring{\gamma}}_{4j} = \hat{\mathring{\gamma}}_{i4} = 0 \tag{3.1.15}$$

and determine dl in the initial state, i.e.

$$d\mathring{l}^2 = \hat{\mathring{\gamma}}_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta = \hat{\mathring{\gamma}}_{ij}\, d\xi^i\, d\xi^j. \tag{3.1.16}$$

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[4] Quantities with an asterisk refer to a reference frame that is proper or the specified point in the medium with coordinates ξ^α_* at time t_*. Such is the inertial reference frame of the observer, with respect to which $v^{*\gamma}(\xi^\alpha_*, t_*) = 0$ and, hence, $u^{*i}(\xi^\alpha_*, t_*) = \delta^i_4$, where δ^i_{j*} is the Kronecker delta. In the Minkowski space-time, each proper reference frame has corresponding to it a proper system of coordinates.

The quantities $\mathring{\rho}$ and $\hat{\mathring{\gamma}}_{ij}$ are, by definition, known functions of the ξ^{α}, i.e. belong to parameters of the type $\hat{K}_B$ (the functions are not varied).

The density ρ of the medium, defined by the first equation in (3.1.12), satisfies the relativistic continuity equation[5]

$$\nabla_i \rho u^i = 0. \tag{3.1.17}$$

From (3.1.13) and (3.1.15) it follows that under a transformation of the comoving system of coordinates

$$\xi'^{\alpha} = \xi'^{\alpha}(\xi^{\beta}), \quad \xi'^4 = \xi'^4(\xi^{\beta}, \xi^4) \tag{3.1.18}$$

the quantities $\hat{\gamma}_{\alpha\beta}$ and $\hat{\mathring{\gamma}}_{\alpha\beta}$ behave like covariant components of three-dimensional tensors, e.g.

$$\hat{\gamma}'_{\alpha\beta} = \frac{\partial \xi^i}{\partial \xi'^{\alpha}} \frac{\partial \xi^j}{\partial \xi'^{\beta}} \hat{\gamma}_{ij} = \frac{\partial \xi^{\delta}}{\partial \xi'^{\alpha}} \frac{\partial \xi^{\nu}}{\partial \xi'^{\beta}} \hat{\gamma}_{\delta\gamma}. \tag{3.1.19}$$

Similar relationships hold for the $\hat{\mathring{\gamma}}_{\alpha\beta}$, with

$$\hat{\gamma}'_{i4} = \hat{\gamma}'_{4j} = 0, \quad \hat{\mathring{\gamma}}'_{i4} = \hat{\mathring{\gamma}}'_{4j} = 0. \tag{3.1.20}$$

For this reason the ratio $\mathring{\gamma}/\hat{\gamma}$ and the density ρ are invariant under transformations (3.1.18) of a comoving system of coordinates. Since by definition the Lagrangian is a four-dimensional scalar, the function U in (3.1.8) must also be invariant under transformations (3.1.18). This implies that for an isotropic body, where all the $\hat{K}_B$ are scalars, the function U may depend on combinations of the components $\hat{\gamma}_{ij}$ and $\hat{\mathring{\gamma}}_{ij}$ that are invariant under transformations (3.1.18). Allowing for (3.1.13) and (3.1.15), we can show that there are only three independent combinations of this type. For instance, they may be taken in the form of the following scalars:

$$\begin{aligned} &\mathring{I}_1 = \hat{\mathring{\gamma}}^{\alpha\beta}\hat{\varepsilon}_{\alpha\beta} \equiv \hat{\mathring{\gamma}}^{ij}\hat{\varepsilon}_{ij}, \quad \mathring{I}_2 = \hat{\mathring{\gamma}}^{\alpha\delta}\hat{\mathring{\gamma}}^{\beta\nu}\hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta} \equiv \hat{\mathring{\gamma}}^{ih}\hat{\mathring{\gamma}}^{jr}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{rh}, \\ &\mathring{I}_3 = \hat{\mathring{\gamma}}^{\alpha\zeta}\hat{\mathring{\gamma}}^{\beta\gamma}\hat{\mathring{\gamma}}^{\delta e}\hat{\varepsilon}_{\alpha\beta}\hat{\varepsilon}_{\gamma\delta}\hat{\varepsilon}_{\varepsilon\zeta} \equiv \hat{\mathring{\gamma}}^{in}\hat{\mathring{\gamma}}^{jr}\hat{\mathring{\gamma}}^{km}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{rk}\hat{\varepsilon}_{mn}, \end{aligned} \tag{3.1.21}$$

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[5] This can easily be verified if we note that $\nabla_i \rho u^i = \hat{\nabla}_i \rho \hat{u}^i =$

$$\frac{1}{\sqrt{-\hat{g}}}\, \hat{\partial}_i \left(\sqrt{-\hat{g}}\, \mathring{\rho} \sqrt{\mathring{\gamma}/\hat{\gamma}}\, \delta^i_4 / \sqrt{\hat{g}_{44}}\right) = \frac{1}{\sqrt{-\hat{g}}} \frac{\partial}{\partial \xi^4} \mathring{\rho} \sqrt{\mathring{\gamma}} = 0.$$

where

$$\hat{\varepsilon}_{ij} \equiv (1/2)\,(\hat{\gamma}_{ij} - \hat{\mathring{\gamma}}_{ij}), \quad \hat{\mathring{\gamma}}^{\alpha\beta} \equiv (1/\mathring{\gamma})\,\partial\mathring{\gamma}/\partial\hat{\mathring{\gamma}}_{\beta\alpha}. \tag{3.1.22}$$

The values of the four-dimensional components $\hat{\mathring{\gamma}}^{i4} = \hat{\mathring{\gamma}}^{4i}$ do not affect the invariants $\mathring{I}_{\alpha}$, since $\hat{\varepsilon}_{i4} = \hat{\varepsilon}_{4i} = 0$. For the sake of definiteness we may assume that in a fixed comoving coordinate system $\hat{\mathring{\gamma}}^{i4} = \hat{\mathring{\gamma}}^{4i} = 0$. The quantities $\hat{\mathring{\gamma}}^{\alpha\beta}$ behave under transformations (3.1.14) of the comoving system of coordinates ξ^i like contravariant components or a three-dimensional tensor:

$$\hat{\mathring{\gamma}}'^{\alpha\beta} = \frac{\partial\xi'^{\alpha}}{\partial\xi^{i}}\,\frac{\partial\xi'^{\beta}}{\partial\xi^{j}}\,\hat{\mathring{\gamma}}^{ij} = \frac{\partial\xi'^{\alpha}}{\partial\xi^{\delta}}\,\frac{\partial\xi'^{\beta}}{\partial\xi^{\gamma}}\,\hat{\mathring{\gamma}}^{\delta\gamma},$$

but

$$\hat{\mathring{\gamma}}'^{k4} = \frac{\partial\xi'^{k}}{\partial\xi^{i}}\,\frac{\partial\xi'^{4}}{\partial\xi^{j}}\,\hat{\mathring{\gamma}}^{ij} = \frac{\partial\xi'^{k}}{\partial\xi^{\alpha}}\,\frac{\partial\xi'^{4}}{\partial\xi^{\beta}}\,\hat{\mathring{\gamma}}^{\alpha\beta} \neq 0.$$

3.2 THE EULER EQUATIONS AND CONDITIONS ON DISCONTINUITIES

We take the functions

$$x^i\,(\xi^j), \quad S\,(\xi^j) \tag{3.2.1}$$

as the independent parameters of the problem. Using the quantities $\hat{\gamma}_{ij}$, $\mathring{\gamma}_{ij}$, ρ, $\hat{K}_B$, and $d\hat{V}_4$ defined above and the results of the theory of variations developed by Sedov [3], we obtain

(1) $$\delta\hat{g}_{ij} = \delta\,(g_{kr}x^k{}_i x^r{}_j) = g_{kr}\,(x^p{}_i\nabla_p\delta x^k)\,x^r{}_j + g_{kr}x^k{}_i\,(x^p{}_j\nabla_p\delta x^r)$$
$$= \hat{\nabla}_i\delta\hat{x}_j + \hat{\nabla}_j\delta\hat{x}_i \equiv \hat{\nabla}_{(i}\delta\hat{x}_{j)},$$

where $\delta\hat{x}_j \equiv x^k{}_j\delta x_k \equiv x^k{}_j\,(g_{ki}\delta x^i)$;

(2) $$\delta\hat{u}_i = \delta\left(\hat{g}_{ij}\,\frac{\delta^j_4}{\sqrt{\hat{g}_{44}}}\right) = \frac{\delta^j_4}{\sqrt{\hat{g}_{44}}}\,\hat{\nabla}_{(i}\delta\hat{x}_{j)} - \hat{g}_{ij}\,\frac{\delta^j_4\hat{\nabla}_{(4}\delta\hat{x}_{4)}}{2\sqrt{\hat{g}_{44}}\,\hat{g}_{44}}$$
$$= \hat{u}^j\hat{\nabla}_{(i}\delta\hat{x}_{j)} - \hat{g}_{ij}\hat{u}^j\hat{u}^p\hat{u}^q\hat{\nabla}_p\delta\hat{x}_q = \hat{u}^p\hat{\nabla}_{(i}\delta\hat{x}_{p)} - \hat{u}_i\hat{u}^p\hat{u}^q\hat{\nabla}_p\delta\hat{x}_q;$$

(3) $$\delta\hat{\gamma}_{ij} = \delta\,(\hat{u}_i\hat{u}_j - \hat{g}_{ij}) = \hat{u}^p\hat{u}_j\hat{\nabla}_{(i}\delta\hat{x}_{p)} + \hat{u}^p\hat{u}_i\hat{\nabla}_{(j}\delta\hat{x}_{p)}$$
$$- \hat{u}_i\hat{u}_j\hat{u}^p\hat{u}^q\hat{\nabla}_{(p}\delta\hat{x}_{q)} - \hat{\nabla}_{(i}\delta\hat{x}_{j)} = \hat{\gamma}^p_j\hat{\nabla}_{(i}\delta\hat{x}_{p)}$$

$$+\hat{u}_i\hat{u}^q(\hat{\nabla}_{(q}\delta\hat{x}_{j)}-\hat{u}_j\hat{u}^p\hat{\nabla}_{(q}\delta\hat{x}_{p)})=\hat{\gamma}^p_j\hat{\nabla}_{(i}\delta\hat{x}_{p)}$$

$$-\hat{u}_i\hat{u}^q\hat{\gamma}^p_j\hat{\nabla}_{(q}\delta\hat{x}_{p)}=-\hat{\gamma}^p_j\hat{\gamma}^q_i\hat{\nabla}_{(q}\delta\hat{x}_{p)}=-\hat{\gamma}^{(q}_i\hat{\gamma}^{p)}_j\hat{\nabla}_p\delta\hat{x}_q;$$

$$(4)\quad \delta\rho=\delta\left(\mathring{\rho}\sqrt{\mathring{\gamma}/\hat{\gamma}}\right)=-\frac{\mathring{\rho}\sqrt{\mathring{\gamma}}}{2\hat{\gamma}\sqrt{\hat{\gamma}}}\frac{\partial\hat{\gamma}}{\partial\hat{\gamma}_{\alpha\beta}}\delta\hat{\gamma}_{\alpha\beta}=-\frac{1}{2}\rho\,\|\hat{\gamma}_{\alpha\beta}\|^{-1}\delta\hat{\gamma}_{\alpha\beta}$$

$$=(1/2)\,\rho\hat{g}^{\alpha\beta}\delta\hat{\gamma}_{\alpha\beta}=-(1/2)\,\rho\hat{g}^{\alpha\beta}\hat{\gamma}^{(i}_\alpha\hat{\gamma}^{j)}_\beta\hat{\nabla}_j\delta\hat{x}_i$$

$$=-(1/2)\,\rho\hat{g}^{pq}\hat{\gamma}^{(i}_p\hat{\gamma}^{j)}_q\hat{\nabla}_j\delta\hat{x}_i=(1/2)\,\rho\hat{\gamma}^{(ij)}\hat{\nabla}_j\delta\hat{x}_i=\rho\hat{\gamma}^{ij}\hat{\nabla}_j\delta\hat{x}_i$$

(in deriving the formula for $\delta\rho$ we used the fact that $\|\hat{\gamma}_{\alpha\beta}\|^{-1}=-\|\hat{g}^{\alpha\beta}\|$, which follows from

$$\hat{g}^{\alpha\beta}\hat{\gamma}_{\beta\delta}=\hat{g}^{\alpha i}\hat{\gamma}_{i\delta}=\hat{g}^{\alpha i}(\hat{u}_i\hat{u}_\delta-\hat{g}_{i\delta})=\hat{u}^\alpha\hat{u}_\delta-\delta^\alpha_\delta=-\delta^\alpha_\delta);$$

$$(5)\quad \delta(dV_4)=\delta\left(\sqrt{-\hat{g}}\,d^4\xi\right)=-\frac{1}{2\sqrt{-\hat{g}}}\frac{\partial\hat{g}}{\partial\hat{g}_{ij}}\delta\hat{g}_{ij}d^4\xi$$

$$=(1/2)\sqrt{-\hat{g}}\,\hat{g}^{ij}\hat{\nabla}_{(i}\delta\hat{x}_{j)}d^4\xi=(\hat{\nabla}_i\delta\hat{x}^i)\,dV_4=(\nabla_i\delta x^i)\,dV_4.$$

Hence,

$$\delta\hat{\gamma}_{ij}=-\hat{\gamma}_{(iq}\hat{\gamma}_{j)p}\xi_r{}^q\xi_k{}^p\nabla^k\delta x^r,\quad \delta\rho=\rho\gamma_{ij}\nabla^j\delta x^i,$$
$$\delta\,dV_4=(\nabla_i\delta x^i)\,dV_4,\quad \delta\mathring{\gamma}_{ij}=0,\quad \delta\hat{K}_B=0, \qquad (3.2.2)$$

where

$$\delta\hat{x}^q=\xi_i{}^q\delta x^i,\quad \xi_i{}^q\equiv\partial\xi^q/\partial x^i.$$

Now in Eq. (3.1.5) we carry out a variation process and employ (3.2.2). This yields

$$\int\limits_{V_4}\{(\nabla^jT_{ij}-Q_i)\,\delta x^i+\rho\,(T-\partial U/\partial S)\,\delta S-\nabla^j(T_{ij}\delta x^i)\}\,dV_4+\delta W=0, \qquad (3.2.3)$$

where

$$T_{ij}=\rho\,(c^2+U)\,u_iu_j-p_{ij}, \qquad (3.2.4)$$

$$p_{ij}=\xi_i{}^k\xi_j{}^r\hat{p}_{kr},\quad \hat{p}_{ij}=2\rho\,(\partial U/\partial\hat{\gamma}_{\alpha\beta})\,\hat{\gamma}_{\alpha i}\hat{\gamma}_{\beta j}. \qquad (3.2.5)$$

In view of the fact that the variations δx^i and δS are arbitrary and the definition of δW, (3.2.4) and (3.2.5) yield the Euler equations and the formula for δW:

$$\nabla^j T_{ij} = Q_i, \quad \partial U/\partial S = T, \tag{3.2.6}$$

$$\delta W = (1/c) \int\limits_{\partial \hat{V}_4 + \hat{\Sigma}_\pm} T_{ij} \delta x^i n^j \, d\Sigma. \tag{3.2.7}$$

Here $\Sigma_\pm$ are two sides of the three-dimensional surface of strong discontinuity inside region V_4, $d\Sigma$ an element of the three-dimensional volume of surface $\partial V_4 + \Sigma_\pm$, n^j the components of a unit 4-vector of an external normal to this surface. Fixing the value of δW at $\delta x^i = 0$ at the boundary ∂V_4, we can find the conditions on the surface of a strong discontinuity. For instance, $\delta W = 0$ when there are no external actions; hence, for variations δx^i continuous at the discontinuity we obtain from (3.2.7) the following formula for δW:

$$(T_{ij})_+ n^j = (T_{ij})_- n^j, \quad n^j \equiv n^j_+ = -n^j_-.$$

The first equation in (3.2.6) is the energy-momentum equation for a perfectly elastic body. The T_{ij} are the components of the total energy-momentum tensor, while the p_{ij} are the components of the four-dimensional internal stress tensor. Equations (3.2.6) retain their shape if we go over to a comoving system of coordinates, provided we "cover" all tensor components with hats.

The definitions (3.2.5) of $\hat{p}_{ij}$, p_{ij} imply that

$$\hat{p}_{\alpha\beta} = 2\rho\,(\partial U/\partial \hat{\gamma}_{\gamma\delta})\, \hat{\gamma}_{\gamma\alpha} \hat{\gamma}_{\delta\beta}, \quad \hat{p}_{i4} = \hat{p}_{4i} = 0, \quad u^k p_{ik} = 0. \tag{3.2.8}$$

Substituting u^k given by (3.1.3) into the third equation in (3.2.8), we find that

$$p_{\alpha 4} = -(v^\beta/c) p_{\alpha\beta}, \quad p_{44} = -(v^\beta/c) p_{4\beta} = (v^\alpha v^\beta/c^2) p_{\alpha\beta}. \tag{3.2.9}$$

In the observer's inertial reference frame, the first equation in (3.2.6) takes the form

$$(1/c)\, \partial\, [\rho\, (c^2 + U) u_k u_4 - p_{k4}]/\partial t - \tilde{\nabla}^\beta [\rho\, (c^2 + U) u_k u_\beta - p_{k\beta}] = Q_k, \tag{3.2.10}$$

where we have allowed for (3.2.4), and $\tilde{\nabla}^\beta$ is the symbol of a contravariant derivative in three-dimensional space with the metric tensor components $h_{\alpha\beta}$. Substituting u_k given by (3.1.3) and p_{k4} given by (3.2.9) into Eq. (3.2.10), we can find, for $i = \alpha$ and $i = 4$, the momentum and energy equations:

$$\partial \left[\rho\, (c^2 + U) \frac{v_\alpha}{c^2 - v^2} - p_{\alpha\beta} \frac{v^\beta}{c^2}\right]/\partial t + \tilde{\nabla}^\beta \left[\rho\, (c^2 + U) \frac{v_\alpha v_\beta}{c^2 - v^2} - p_{\alpha\beta}\right] = h_{\alpha\beta} Q^\beta, \tag{3.2.11}$$

$$\partial \left[\rho\, (c^2 + U) \frac{c^2}{c^2 - v^2} - p_{\alpha\beta} \frac{v^\alpha v^\beta}{c^2}\right]/\partial t + \tilde{\nabla}^\beta \left[\rho\, (c^2 + U) \frac{v_\beta c^2}{c^2 - v^2} - p_{\alpha\beta} v^\alpha\right] = cQ_4, \tag{3.2.12}$$

where according to (3.2.2) and (3.2.8)

$$p_{\alpha\beta} = \hat{p}_{ij}\,(\partial\xi^i/\partial x^\alpha)\,(\partial\xi^j/\partial x^\beta) = \hat{p}_{\gamma\delta}\,(\partial\xi^\gamma/\partial x^\alpha)_t\,(\partial\xi^\delta/\partial x^\beta)_t,$$
$$\hat{p}_{\gamma\delta} = 2\rho\,(\partial U/\partial\hat{\gamma}_{\alpha\beta})\,\hat{\gamma}_{\alpha\gamma}\hat{\gamma}_{\beta\delta}. \tag{3.2.13}$$

Let us also write in three-dimensional form the continuity equation (3.1.17):

$$\partial\left(\rho/\sqrt{c^2-v^2}\right)/\partial t + \nabla_\alpha\left(\rho v^\alpha/\sqrt{c^2-v^2}\right) = 0. \tag{3.2.14}$$

If at a fixed moment in time for a fixed infinitesimal particle of the continuous medium the observer's reference frame is the proper system, then Eqs. (3.2.14), (3.2.11), and (3.2.12) simplify considerably:

$$\partial\rho/\partial t^* + \rho\tilde{\nabla}^*_\alpha v^{*\alpha} = 0, \tag{3.2.15}$$

$$[\rho\,(1+U/c^2)\,h^*_{\alpha\beta} - (1/c^2)\,p^*_{\alpha\beta}]\,\partial v^{*\beta}/\partial t^* = \tilde{\nabla}^{*\beta} p^*_{\alpha\beta} + h^*_{\alpha\beta}Q^{*\beta}, \tag{3.2.16}$$

$$\rho\,(\partial U/\partial t^*) = p^*_{\alpha\beta}\tilde{\nabla}^{*\beta}v^{*\alpha} + cQ^*_4, \quad cQ^*_4 = cQ_i u^i = dQ^{(e)}/d\tau = dQ^{(e)}/dt^*, \tag{3.2.17}$$

with

$$p^*_{\alpha\beta} = 2\rho\,(\partial U/\partial\hat{\gamma}_{\gamma\delta})\,(\partial x^{*\mu}/\partial\xi^\gamma)_{t^*}\,(\partial x^{*\nu}/\partial\xi^\delta)_{t^*}\,h^*_{\mu\alpha}h^*_{\nu\beta}, \quad p^*_{i4} = p^*_{4i} = 0, \tag{3.2.18}$$

$$\hat{\gamma}_{\alpha\beta} = h^*_{\gamma\delta}\,(\partial x^{*\gamma}/\partial\xi^\alpha)_{t^*}\,(\partial x^{*\delta}/\partial\xi^\beta)_{t^*}, \quad \gamma^*_{\alpha\beta} = h^*_{\alpha\beta}, \tag{3.2.19}$$

$$T^*_{\alpha\beta} = p^*_{\alpha\beta}, \quad T^*_{\alpha 4} = T^*_{4\alpha} = 0, \quad T^*_{44} = \rho\,(c^2+U). \tag{3.2.20}$$

The problem of establishing the laws of motion in an arbitrary fixed reference frame of the observer by the known laws of motion in the proper reference frame belongs to navigation theory. The transition to an arbitrary inertial reference frame (the observer's frame) from the medium's proper reference frame, where $g^*_{ij} = g_{ij}$, is realized via a Lorentz transorfmation. As a result in an arbitrary inertial reference frame of the observer the relativistic equations of the model of a perfectly elastic body, (3.1.17) and (3.2.6), can be put in the following form

$$\nabla_j\,(\rho d_i{}^j u^{*i}) = 0, \quad \nabla^j\,(b_i{}^k b_j{}^r T^*_{kr}) = b_i{}^k Q^*_k, \tag{3.2.21}$$

where $d_i{}^j$ are the matrix elements of the transformation that connects the basis vectors in the observer's system of coordinates, Э_j, with the basis vectors Э^*_i in the proper system of coordinates:

$$\text{Э}^*_i = d_i{}^j\text{Э}_j, \tag{3.2.22}$$

and $b_i{}^k$ are the elements of the matrix inverse to $\| d_i{}^j \|$. When both bases are orthonormal and their spatial counterparts, Э^*_α and Э_β, are oriented in the three-dimensional physical space in a like manner, the matrix elements $d_i{}^j$ have a form indicated by Sedov [4].

3.3 TRANSITION TO NEWTONIAN MECHANICS

Equations (3.2.15)-(3.2.17) differ from the equations for the model of a perfectly elastic body in Newtonian mechanics written in the proper reference frame only in terms with coefficients c^{-2}. If $U = 0$ (this is the case for dust, for one), then $p_{\alpha\beta} = 0$ and the above-mentioned terms cease to exist. In this case the equations of relativistic mechanics coincide with those of Newtonian mechanics (in the proper reference frame). But in an arbitrary inertial reference frame they differ since the former are obtained via a navigational calculation using the Lorentz transformation, while the latter are obtained by solving a navigational problem with Galilean transformations. The equations of relativistic and Newtonian mechanics for arbitrary inertial reference frames prove to be invariant under Lorentz and Galilean transformations, respectively. If $U \neq 0$, Eqs. (3.2.15)-(3.2.17) with terms proportional to c^{-2} can be considered within the framework of Newtonian mechanics as laws that determine the motion (in the proper reference frame) of a perfectly elastic body with more complex properties. To obtain the respective laws in an arbitrary inertial reference frame, we must solve a navigational problem using Galilean transformations. As a result we arrive at a set of equations, (3.3.1)-(3.3.3), that are invariant under Galilean transformations; in deriving Eq. (3.3.3) from (3.2.17) we must take into account Eqs. (3.2.15) and (3.2.16). Thus,

$$\partial\tilde{\rho}/\partial t + \tilde{\nabla}_\alpha \tilde{\rho} v^\alpha = 0, \tag{3.3.1}$$

$$\tilde{\rho}\frac{dv_\alpha}{dt} + \frac{1}{c^2}(\tilde{\rho}\tilde{U}h_{\alpha\beta} - \tilde{p}_{\alpha\beta})\frac{dv^\beta}{dt} = \tilde{\nabla}^\beta \tilde{p}_{\alpha\beta} + \tilde{Q}_\alpha, \tag{3.3.2}$$

$$\rho\frac{d}{dt}\left(\frac{v^2}{2} + \tilde{U}\right) + \frac{1}{c^2}(\tilde{\rho}\tilde{U}h_{\alpha\beta} - \tilde{p}_{\alpha\beta})v^\alpha\frac{dv^\beta}{dt} = \tilde{\nabla}^\beta(\tilde{p}_{\alpha\beta}v^\alpha) + \frac{dQ^{(e)}}{dt} + \tilde{Q}_\alpha v^\alpha, \tag{3.3.3}$$

where

$$\tilde{\rho} = \mathring{\rho}\sqrt{\mathring{\gamma}/\hat{h}}, \quad \tilde{U} = U\,|_{\hat{\gamma}_{\alpha\beta} = \hat{h}_{\alpha\beta}}, \quad \tilde{Q}_\alpha = h_{\alpha\beta}(\partial x^\beta/\partial x^{*\gamma})\,Q^{*\gamma}, \tag{3.3.4}$$

$$\hat{h}_{\alpha\beta} = h_{\gamma\delta}(\partial x^\gamma/\partial\xi^\alpha)_t(\partial x^\delta/\partial\xi^\beta)_t, \quad \hat{h} = |\,\hat{h}_{\alpha\beta}\,|, \tag{3.3.5}$$

$$\hat{p}_{\alpha\beta} = 2\tilde{\rho}\,(\partial\tilde{U}/\partial\hat{h}_{\gamma\delta})(\partial x^\mu/\partial\xi^\gamma)_t(\partial x^\nu/\partial\xi^\delta)_t\,h_{\mu\alpha}h_{\nu\beta}. \tag{3.3.6}$$

If the observer's system of coordinates x^i coincides with the proper coordinate system x^{*i}, on the basis of (3.2.19) we have

$$\hat{h}_{\alpha\beta}\,|_{x^i=x^{*i}}=\hat{\gamma}_{\alpha\beta},\quad \tilde{\rho}\,|_{x^i=x^{*i}}=\rho, \tag{3.3.7}$$

$$\tilde{U}\,|_{x^i=x^{*i}}=U,\quad \hat{p}_{\alpha\beta}\,|_{x^i=x^{*i}}=p^*_{\alpha\beta}, \tag{3.3.8}$$

and, hence, Eqs. (3.3.1)-(3.3.3) in the proper reference frame indeed coincide with Eqs. (3.2.15)-(3.2.17).

Equations (3.3.1)-(3.3.3) can obviously be used to describe within the scope of Newtonian mechanics the motion of a continuous medium when in the observer's reference frame

$$\varepsilon^2 \equiv \max\,(v^2/c^2,\ vx_0/c^2t_0) \ll 1,$$

where x_0 and t_0 are characteristic values of distance and time. The Lorentz transformation is reduced to a Galilean transformation if we neglect terms of the order ε^2; hence, Eqs. (3.3.1)-(3.3.3) differ from the relativistic equations (3.2.14), (3.2.11), and (3.2.12) in terms of the order of ε^2. The fact that Eqs. (3.3.2) and (3.3.3) contain terms proportional to c^{-2} implies that the kinematic condition $\varepsilon^2 \ll 1$ is insufficient for a transition to the model equations of a perfectly elastic body of Newtonian mechanics. What is needed are additional restrictions of a dynamic nature that ensure the smallness of these terms. If only condition $\varepsilon^2 \ll 1$ is met, then terms in Eqs. (3.3.2) and (3.3.3) proportional to c^{-2} may be important in Newtonian mechanics, which leads to a model of a perfectly elastic body with more complex properties caused by internal (i.e. existing in the proper reference frame) relativistic effects.

Equations (3.3.2) and (3.3.3.) may also be obtained if we start from the variational equation (3.1.5) written in three-dimensional form (in a synchronous inertial reference frame):

$$\tilde{\delta}\int_{t_1}^{t_2}\int_{V_3}\tilde{\Lambda}\,dV_3\,dt+\delta W^*+\delta W=0, \tag{3.3.9}$$

where $dV_3=\sqrt{h}\,dx^1\,dx^2\,dx^3=\sqrt{\hat{h}}\,d\xi^1\,d\xi^2\,d\xi^3$,

$$\tilde{\Lambda}=-\tilde{\rho}c^2-\tilde{\rho}\tilde{U}\,(S,\ \hat{h}_{\alpha\beta},\ \hat{\mathring{\gamma}}_{\alpha\beta},\ \hat{K}_B), \tag{3.3.10}$$

$$\delta W^*=\int_{t_1}^{t_2}\int_{V}\Big[\tilde{\rho}T\delta S+(\tilde{Q}_\beta-\rho_{\alpha\beta}a^\alpha)\,\tilde{\delta}x^\beta$$

$$+\left(1\Big/\sqrt{\hat{h}}\right)d\left(\sqrt{\hat{h}}\,\rho_{\alpha\beta}v^\alpha\tilde{\delta}x^\beta\right)\Big/\,dt\Big]\,dV_3\,dt, \tag{3.3.11}$$

$$\rho_{\alpha\beta}\equiv\tilde{\rho}\,(1+U/c^2)h_{\alpha\beta}-(1/c^2)\tilde{p}_{\alpha\beta},\quad a^\alpha\equiv dv^\alpha/dt, \tag{3.3.12}$$

$$\rho T\,(dS/dt)=dQ^{(e)}/dt. \tag{3.3.13}$$

(In Eq. (3.3.9) the variation $\tilde{\delta}$ is taken at constant ξ^α and t.)

If $\widetilde{U} \equiv 0$ or $c = \infty$, then

$$\rho_{\alpha\beta} = \tilde{\rho} h_{\alpha\beta} \tag{3.3.14}$$

and, hence,

$$\delta W^* = \int_{t_1}^{t_2} \int_{V_3} [\tilde{\rho} T \tilde{\delta} S + Q_\beta \tilde{\delta} x^\beta] \, dV_3 \, dt + \tilde{\delta} \int_{t_1}^{t_2} \int_{V_3} (\tilde{\rho} v^2/2) \, dV_3 \, dt. \tag{3.3.15}$$

If we wish to go over to an arbitrary fixed and generally deformable noninertial reference frame, then for acceleration $a^\alpha \text{Э}_\alpha$ with respect to an inertial reference frame we must use the general formula derived by Sedov [5]:

$$\begin{gathered} a^\alpha \text{Э}_\alpha = \mathbf{a}_{(\mathrm{r})} + \mathbf{a}_{(\mathrm{t})} + 2 v^\alpha_{(\mathrm{r})} (e_{\alpha\beta} + \omega_{\alpha\beta}) \, h^{\beta\gamma} \text{Э}_\gamma, \\ e_{\alpha\beta} = (1/2) (\nabla_\alpha v_{(\mathrm{t})\beta} + \nabla_\beta v_{(\mathrm{t})\alpha}), \quad \omega_{\alpha\beta} = (1/2) (\nabla_\alpha v_{(\mathrm{t})\beta} - \nabla_\beta v_{(\mathrm{t})\alpha}). \end{gathered} \tag{3.3.16}$$

Here $\mathbf{v}_{(\mathrm{r})}$ and $\mathbf{a}_{(\mathrm{r})}$ are the relative velocity and acceleration in a noninertial reference frame, and $\mathbf{v}_{(\mathrm{t})}$ and $\mathbf{a}_{(\mathrm{t})}$ are the transport velocity and acceleration of a given point of this reference frame.

Even in the absence of external forces and heat fluxes ($\widetilde{Q}_\alpha = dQ^{(e)} = 0$), Eqs. (3.3.2) and (3.3.3.) are not of the divergence type, and at $\varepsilon^2 \ll 1$ they differ from the divergence equations (3.2.11) and (3.2.12) by terms of the order of ε^2. This can be explained by the fact that in going over from Eqs. (3.2.16) and (3.2.17) written in the proper reference frame to their counterpart in an arbitrary inertial reference frame of the observer we use the Galilean transformation. It is also an indication that the Galilean transformation gives an inaccurate description of the physical space-time.

The above reasoning concerning the model of a perfectly elastic body can be generalized to arbitrary models, since in their very essence physical laws formulated in the proper reference frame reflect the internal processes inherent in particles of a continuous medium. These laws are independent of the arbitrary choice of the observer's reference frame and may be studied in both special relativity and Newtonian mechanics. To obtain the respective laws in the (arbitrary) fixed reference frame of the observer, we need only solve the appropriate navigational problem and employ certain assumptions about space and time (e.g. in special relativity we must use the Lorentz transformation and in Newtonian mechanics the Galilean).

3.4 IRREVERSIBLE PROCESSES

We now consider irreversible processes associated with the irreversible effects of heat conductivity, viscosity and plasticity, and also with the presence of internal degrees of freedom characterized by additional parameters $\hat{\mu}^A$ (these are included in the arguments of the expression for the internal energy). We fix the function U and the functional δW^* in the following manner:

$$U = U(S, \hat{\gamma}_{ij}, \hat{\gamma}^{(\mathrm{p})}_{ij}, \hat{\mu}^A, \hat{\overset{\circ}{\gamma}}_{ij}, \hat{K}_B), \tag{3.4.1}$$

$$\delta W^* = (1/c) \int\limits_{V_4} (\rho T \delta S - Q_i \delta x^i + \tilde{\tau}_{ij} \nabla^i \delta x^j - \hat{M}_A \delta \hat{\mu}^A - \hat{R}^{ij} \delta \hat{\gamma}^{(\mathrm{p})}_{ij})\, dV_4. \tag{3.4.2}$$

Here $\hat{\gamma}^{(\mathrm{p})}_{ij}$ (ξ^i) are the components of the spatial symmetric metric tensor in the (imagined) unloaded state of an individual particle with Lagrangian coordinates ξ^α. These components may vary depending on ξ^4, satisfy the following relationships

$$\hat{u}^i \hat{\gamma}^{(\mathrm{p})}_{ij} = u^i \gamma^{(\mathrm{p})}_{ij} = 0, \quad \gamma^{(\mathrm{p})*}_{4j} = \gamma^{(\mathrm{p})*}_{i4} = 0, \quad \hat{\gamma}^{(\mathrm{p})}_{4j} = \hat{\gamma}^{(\mathrm{p})}_{i4} = 0, \tag{3.4.3}$$

and determine the value of dl in a state corresponding to total unloading of an individual infinitesimal particle of the medium:

$$dl^{(\mathrm{p})2} = \hat{\gamma}^{(\mathrm{p})}_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta = \hat{\gamma}^{(\mathrm{p})}_{ij}\, d\xi^i\, d\xi^j = \gamma^{(\mathrm{p})}_{ij}\, dx^i\, dx^j. \tag{3.4.4}$$

Generally speaking, the components $\hat{\gamma}^{(\mathrm{p})}_{ij}$ are the sought varied functions of ξ^i. For a perfectly elastic body, in which all deformation processes are reversible by definition, these components are known functions of the Lagrangian coordinates ξ^α and do not change along the world lines of the medium. Then we can consider the $\hat{\gamma}^{(\mathrm{p})}_{ij}$ to be unvaried functions of the type $\hat{K}_B$ (ξ^α), and the common notation is $\hat{\gamma}^{(\mathrm{p})}_{ij} \equiv \hat{\overset{\circ}{\gamma}}_{ij}$.

The components $\tilde{\tau}_{ij}$, $\hat{M}_A$, and $\hat{R}^{ij}$ have been introduced into (3.4.2) to account for irreversible processes, and without loss of generality we may assume that

$$\hat{R}^{ij} = \hat{R}^{ji}, \quad \hat{R}^{ij} \hat{u}_i = \hat{R}^{ij} \hat{u}_j = 0. \tag{3.4.5}$$

Indeed, if we allow for the symmetry of the components $\hat{\gamma}^{(\mathrm{p})}_{ij}$ and for (3.4.3), where the latter implies that

$$\hat{u}^i \delta \hat{\gamma}^{(\mathrm{p})}_{ij} = \hat{u}^j \delta \hat{\gamma}^{(\mathrm{p})}_{ij} = 0, \tag{3.4.6}$$

then we can always transform the last term in the integrand in (3.4.2) in such a way that the coefficients $\hat{R}^{ij}$ satisfy conditions (3.4.5).

When the function U and the functional δW^* are given by (3.4.1) and (3.4.2), the variational equation (3.1.5) leads to the following expression for δW:

$$\delta W = (1/c) \int\limits_{\partial V_4 + \Sigma_\pm} T_{ij}\delta x^i n^j \, d\Sigma, \tag{3.4.7}$$

where

$$T_{ij} = \rho\,(c^2 + U)\,u_i u_j - p_{ij} - \tilde{\tau}_{ij}, \tag{3.4.8}$$

and the Euler equations[6]

$$\begin{gathered} \nabla_j T^j_i = Q_i, \quad \partial U/\partial S = T, \\ -\rho\,(\partial U/\partial \hat{\gamma}^{(\mathrm{p})}_{pq})\,\hat{\gamma}^i_p \hat{\gamma}^j_q = \hat{R}^{ij}, \quad -\rho\,\partial U/\partial \hat{\mu}^A = \hat{M}_A. \end{gathered} \tag{3.4.9}$$

To complete the set of equations (3.4.5) and (3.4.9) we need only provide the laws determining the quantities Q_i, $\tilde{\tau}_{ij}$, $\hat{M}_A$, and $\hat{R}^{ij}$. To this end we consider the entropy balance equation and the expression for the dissipation functional, using in this connection the theoretical hypotheses that are generalizations of the hypotheses used in the thermodynamics of irreversible processes [6].

Multiplying the first equation in (3.4.9) into u^i and allowing for the other equations, we obtain

$$\rho T\,(dS/d\tau) = c\,(\nabla_j \tilde{\tau}^{ij})\,u_i + cQ_i u^i + \hat{M}_A\,(d\hat{\mu}^A/d\tau) + \hat{R}^{ij}\,(d\hat{\gamma}^{(\mathrm{p})}_{ij}/d\tau). \tag{3.4.10}$$

If $Q_i u^i = 0$ and the processes are reversible ($\hat{\tau}^{ij} = \hat{M}_A = d\hat{\gamma}^{(\mathrm{p})}_{ij}/d\tau = 0$), then (3.4.10) reflects the fact that all processes are adiabatic. If the processes are reversible but $dQ^{(\mathrm{e})} \neq 0$, then $Q_i u^i$ is also nonzero, with

$$cQ_i u^i = dQ^{(\mathrm{e})}/d\tau. \tag{3.4.11}$$

This may be the case when there is an influx of energy to a perfectly elastic body. For one, such an influx may be due to absorption by the body of the electromagnetic energy of radiation which then turns into heat inside the body; whence the relationship at $dQ^{(\mathrm{e})} \neq 0$. In this connection we must note that the effect of the radiation on the body is connected with the 4-vector force $Q_i \text{Э}^i$, whose work determines in this model the heat influx.

We will restrict our discussion to models in which the components T_{ij} of the energy-momentum tensor are symmetric. Then from the definition (3.4.8) of these components and the fact that the p_{ij} are symmetric (see their definition in (3.2.5)) it follows that the components $\tilde{\tau}_{ij}$ must also be symmetric. We fix them in the form

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[6] Here the relationship (3.4.6) is accounted for by Lagrange's method of multipliers, after determining which we arrive at the third equation in (3.4.9).

$$\begin{aligned}&\tilde{\tau}^{ij} = \tau^{ij} - (1/c)q^{(i}u^{j)}, \quad \tau^{ij}u_i = \tau^{ij}u_j = 0,\\ &u_i q^i = 0, \quad \tau^{ij} = \tau^{ji},\end{aligned} \tag{3.4.12}$$

where τ^{ij} are the components of the four-dimensional viscous stress tensor, and q^i the components of the heat flux 4-vector.

We can now easily write Eq. (3.4.10) in the form

$$\begin{aligned}&\rho\frac{dS}{d\tau} \equiv \rho\frac{d_eS}{d\tau} + \rho\frac{d_iS}{d\tau} = -\nabla_i\left(\frac{q^i}{T}\right) + cQ_iu^i + \sigma,\\ &\rho\frac{d_iS}{d\tau} = \sigma \equiv -\frac{q^i}{T^2}(\nabla_iT - Tu^j\nabla_ju_i) - \frac{1}{2}\frac{\tau^{ij}}{T}\nabla_j(cu_i)\\ &+\frac{\hat{R}^{ij}}{T}\frac{d\hat{\gamma}^{(\mathrm{p})}_{ij}}{a\tau} + \frac{\hat{M}_A}{T}\frac{d\hat{\mu}^A}{d\tau}.\end{aligned} \tag{3.4.13}$$

It can be shown that if we allow for conditions (3.4.3) and (3.4.12) and the fact that $\hat{\varepsilon}_{4\alpha} = \hat{\varepsilon}_{\alpha 4} = 0$, $\hat{\gamma}^i_4 = 0$, and $\hat{w}_4 = 0$, with $\hat{w}_i \equiv \hat{u}^j\hat{\nabla}_j\hat{u}_i$ the components of the 4-vector acceleration of the continuous medium, then the three first terms in the second equation in (3.4.13) can be transformed in the following way:

$$-\frac{q^i}{T^2}(\nabla_iT - Tw_i) = \frac{q^i}{T^2}(\gamma^j_i\nabla_jT + Tw_i) = \frac{\hat{q}^\alpha}{T^2}(\hat{\gamma}^j_\alpha\hat{\nabla}_jT + T\hat{w}_\alpha),$$

$$\frac{-(1/2)\,\tau^{ij}}{T}\nabla_{(i}cu_{j)} = \frac{(1/2)\,\hat{\tau}^{ij}}{T}\frac{d\hat{\gamma}_{ij}}{d\tau} = \frac{\hat{\tau}^{ij}}{T}\frac{d\hat{\varepsilon}_{ij}}{d\tau} = \frac{\hat{\tau}^{\alpha\beta}}{T}\frac{d\hat{\varepsilon}_{\alpha\beta}}{d\tau},$$

$$\frac{\hat{R}^{ij}}{T}\frac{d\hat{\gamma}^{(\mathrm{p})}_{ij}}{d\tau} = \frac{\hat{R}^{\alpha\beta}}{T}\frac{d\hat{\gamma}^{(\mathrm{p})}_{\alpha\beta}}{d\tau}.$$

After this the expression for the dissipation function in (3.4.13) assumes the form

$$\sigma = \frac{q^\alpha}{T^2}(\hat{\gamma}^j_\alpha\hat{\nabla}_jT + T\hat{w}_\alpha) + \frac{\hat{\tau}^{\alpha\beta}}{T}\frac{d\hat{\varepsilon}_{\alpha\beta}}{d\tau} + \frac{\hat{R}^{\alpha\beta}}{T}\frac{d\hat{\gamma}^{(\mathrm{p})}_{\alpha\beta}}{d\tau} + \frac{\hat{M}_A}{T}\frac{d\hat{\mu}_A}{d\tau}. \tag{3.4.14}$$

To fix the generalized fluxes $\hat{q}^\alpha/T^2$, $\hat{\tau}^{\alpha\beta}/T$, $\hat{R}^{\alpha\beta}/T$, and $\hat{M}_A/T$ in the case where they are functions of the generalized forces

$$\hat{\theta}_\alpha \equiv \hat{\gamma}^j_\alpha\hat{\nabla}_jT + T\hat{w}_\alpha, \quad d\hat{\varepsilon}_{\alpha\beta}/d\tau, \quad d\hat{\gamma}^{(\mathrm{p})}_{\alpha\beta}/d\tau, \quad d\hat{\mu}^A/d\tau, \tag{3.4.15}$$

we may take the following generalized laws [6]:

$$\begin{aligned}&(1/T^2)\,\hat{q}^\alpha = \lambda_1(\partial\sigma/\partial\hat{\theta}_\alpha), \quad \hat{\tau}^{\alpha\beta}/T = \lambda_2[\partial\sigma/\partial(d\hat{\varepsilon}_{\alpha\beta}/d\tau)],\\ &\hat{R}^{\alpha\beta}/T = \lambda_3[\partial\sigma/\partial(d\hat{\gamma}^{(\mathrm{p})}_{\alpha\beta}/d\tau)], \quad \hat{M}_A/T = \lambda_4[\partial\sigma/\partial(d\hat{\mu}^A/d\tau)].\end{aligned} \tag{3.4.16}$$

Here σ is a given function of the generalized forces (3.4.15), while the multipliers λ_1, λ_2, λ_3, and λ_4 are defined in such a way that the relationship (3.4.14) for σ is satisfied identically.

If no irreversible crossed processes exist, then

$$\sigma = \sigma_1(\hat\theta_\alpha) + \sigma_2(d\hat\varepsilon_{\alpha\beta}/d\tau) + \sigma_3(d\hat\gamma^{(p)}_{\alpha\beta}/d\tau) + \sigma_4(d\hat\mu^A_a/d\tau)$$

and the multipliers $\lambda_1, \ldots, \lambda_4$ are determined from the following conditions:

$$\begin{aligned}
&\sigma_1 = \lambda_1(\partial\sigma_1/\partial\hat\theta_\alpha)\,\hat\theta_\alpha, \quad \sigma_2 = \lambda_2[\partial\sigma_2/\partial(d\hat\varepsilon_{\alpha\beta}/d\tau)]\,(d\hat\varepsilon_{\alpha\beta}/d\tau),\\
&\sigma_3 = \lambda_3[\partial\sigma_3/\partial(d\hat\gamma^{(p)}_{\alpha\beta}/d\tau)]\,(d\hat\gamma^{(p)}_{\alpha\beta}/d\tau), \qquad (3.4.17)\\
&\sigma_4 = \lambda_4[\partial\sigma_4/\partial(d\hat\mu^A_a/d\tau)]\,(d\hat\mu^A/d\tau).
\end{aligned}$$

In a medium with linear heat conductivity and viscosity, the functions σ_1 and σ_2 are quadratic functions of their arguments (with $\lambda_1 = \lambda_2 = 1/2$). But to describe irreversible effects of plastic deformations, σ_3 must be a homogeneous function of degree 1 in $d\hat\gamma^{(p)}_{\alpha\beta}/d\tau$ (with $\lambda_3 = 1$) (see [6]).

Employing the fact that the components q^i, τ^{ij}, and R^{ij} of the velocity 4-vector are orthogonal, we can easily write the first three relationships (3.4.16) in four-dimensional form:

$$\begin{aligned}
&(1/T^2)\,\hat q^i = -\lambda_1\hat\gamma^i_\alpha(\partial\sigma/\partial\hat\theta_\alpha) = -\lambda_1\hat\gamma^i_j(\partial\sigma/\partial\hat\theta_j),\\
&\hat\tau^{ij}/T = \lambda_2\hat\gamma^i_\alpha\hat\gamma^j_\beta[\partial\sigma/\partial(d\hat\varepsilon_{\alpha\beta}/d\tau)] = \lambda_2\hat\gamma^i_k\hat\gamma^j_r(\partial\sigma/\partial\hat e_{kr}), \qquad (3.4.18)\\
&\hat R^{ij}/T = \lambda_3\hat\gamma^i_\alpha\hat\gamma^j_\beta[\partial\sigma/\partial(d\hat\gamma^{(p)}_{\alpha\beta}/d\tau)] = (1/2)\,\lambda_3\hat\gamma^i_k\hat\gamma^j_r(\partial\sigma/\partial\hat e^{(p)}_{kr}).
\end{aligned}$$

Here $\hat e_{kr}$ and $\hat e^{(p)}_{kr}$ are the components of the four-dimensional tensors of the rates of total and residual (plastic) deformations:

$$\begin{aligned}
&\hat e_{kr} = d\hat\varepsilon_{kr}/d\tau = (1/2)\,(d\hat\gamma_{kr}/d\tau) = -(1/2)\,\hat\gamma^i_k\hat\gamma^j_r\tilde\nabla_{(j}c\hat u_{i)},\\
&\hat e^{(p)}_{kr} = (1/2)\,(d\hat\gamma^{(p)}_{kr}/d\tau).
\end{aligned} \qquad (3.4.19)$$

If we write the second and third equations in (3.4.18) ($i, j = 1, 2, 3$) in the proper reference frame, after we exclude the $\hat R^{ij}$ via the third equation in (3.4.9) these equations coincide with the laws of viscosity and plasticity in Newtonian mechanics in the absence of crossed effects [6]. If we turn to the first equation in (3.4.18), we see that in the proper reference frame it differs from the law of heat conduction of Newtonian mechanics because the expression for θ^*_α contains the term $Tw^*_\alpha \equiv -(1/c^2)T\,(\partial v^*_\alpha/\partial t)$ caused by relativistic effects; in the case of a medium with linear heat conductivity the equation assumes the form

$$q^{*\beta} = -\varkappa^{*\beta\alpha}\,[\nabla^{*}_{\alpha}T + (T/c^2)\,(\partial v^{*}_{\alpha}/\partial t^{*})], \tag{3.4.20}$$

where $\varkappa^{\beta\alpha}$ are the components of the three-dimensional heat conductivity tensor.[7] Hence, if we go over to Newtonian mechanics and by applying the Galilean transformation write Eqs. (3.4.18) for arbitrary reference frame of the observer with the $\hat{R}^{ij}$ excluded via the third equation in (3.4.9) (note that Eqs. (3.4.18) must be written in the proper reference frame), we arrive at the common laws for viscosity and plasticity [6] and the following law of heat conductivity (for a medium with linear heat conductivity):

$$\begin{aligned}&\tilde{q}^{\beta} = -\tilde{\varkappa}^{\beta\alpha}\,[\tilde{\nabla}_{\alpha}T + (T/c^2)(dv_{\alpha}/dt)],\\&\tilde{\varkappa}^{\beta\alpha} = \varkappa^{*\gamma\delta}\,(\partial\tilde{x}^{\beta}/\partial x^{*\gamma})\,(\partial\tilde{x}^{\alpha}/\partial x^{*\delta}).\end{aligned} \tag{3.4.21}$$

The presence of irreversible processes leads also to new terms in the energy and momentum equations, which are obtained from the energy-momentum relativistic equation when we go over to Newtonian mechanics. To do this we must substitute the expressions for the components $\hat{T}_{ij}$ into the first equation in (3.4.9) and write this equation for the proper reference frame, allowing for the fact that (3.4.12) and (3.1.3) yield

$$\tau_{\alpha 4} = -(v^{\beta}/c)\tau_{\alpha\beta},\quad \tau_{44} = (v^{\alpha}v^{\beta}/c^2)\tau_{\alpha\beta},\quad q_4 = -(v^{\beta}/c)q_{\beta}. \tag{3.4.22}$$

As a result, the first equation in (3.4.9) at $i = \alpha$ and $i = 4$ takes the form

$$\begin{aligned}&[\rho\,(1 + U/c^2)\,h^{*}_{\alpha\beta} - (1/c^2)\,(p^{*}_{\alpha\beta} + \tau^{*}_{\alpha\beta})]\,(\partial v^{*\beta}/\partial t^{*}) + (h^{*}_{\alpha\beta}/c^2)\,(\partial q^{*\beta}/\partial t^{*})\\&= \tilde{\nabla}^{*\beta}\,(p^{*}_{\alpha\beta} + \tau^{*}_{\alpha\beta}) - (h^{*}_{\alpha\gamma}/c^2)\,q^{*(\gamma}\tilde{\nabla}^{*}_{\beta}v^{*\beta)} + h^{*}_{\alpha\beta}Q^{*\beta},\end{aligned} \tag{3.4.23}$$

$$\begin{aligned}&\rho\,(\partial U/\partial t^{*}) + (2q^{*\alpha}/c^2)\,(\partial v^{*}_{\alpha}/\partial t^{*})\\&= (p^{*}_{\alpha\beta} + \tau^{*}_{\alpha\beta})\,\tilde{\nabla}^{*\beta}v^{*\alpha} - \tilde{\nabla}^{*}_{\alpha}q^{*\alpha} + cQ^{*}_{4}.\end{aligned} \tag{3.4.24}$$

If we consider Eqs. (3.4.23) and (3.4.24) within the scope of Newtonian mechanics as the energy and momentum equations for a continuous medium with complex properties written for the proper reference frame and use the Galilean transformation when going over to an arbitrary inertial reference frame of the observer, we arrive at the following equations, which are invariant under such transformations:

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[7] At $i = 4$ (or $j = 4$) and in the proper reference frame, Eqs. (3.4.18) become, after we exclude the $\hat{R}^{ij}$ via the third equation in (3.4.9), trivial identities of the type $0 \equiv 0$.

$$\tilde{\rho}\frac{dv_\alpha}{dt}+\frac{1}{c^2}\left[(\tilde{\rho}\tilde{U}h_{\alpha\beta}-\tilde{p}_{\alpha\beta}-\tilde{\tau}_{\alpha\beta})\frac{dv^\beta}{dt}+\frac{h_{\alpha\beta}}{c^2}\frac{d\tilde{q}^\beta}{dt}\right]$$

$$=\tilde{\nabla}^{*\beta}(\tilde{p}_{\alpha\beta}+\tilde{\tau}_{\alpha\beta})-\frac{h_{\alpha\gamma}}{c^2}\tilde{q}^{(\gamma}\tilde{\nabla}_\beta v^{\beta)}+\tilde{Q}_\alpha, \tag{3.4.25}$$

$$\tilde{\rho}\frac{d}{dt}\left(\frac{v^2}{2}+\tilde{U}\right)+\frac{v^\alpha}{c^2}\left[(\tilde{\rho}\tilde{U}h_{\alpha\beta}-\tilde{p}_{\alpha\beta}-\tilde{\tau}_{\alpha\beta})\frac{dv^\beta}{dt}+\frac{h_{\alpha\beta}}{c^2}\frac{d\tilde{q}^\beta}{dt}\right]$$

$$+\frac{2\tilde{q}^\alpha}{c^2}\frac{dv_\alpha}{dt}=\tilde{\nabla}_\beta[(\tilde{p}^{\alpha\beta}+\tilde{\tau}^{\alpha\beta})v_\alpha-\tilde{q}^\beta]$$

$$-\frac{v_\gamma}{c^2}\tilde{q}^{(\gamma}\tilde{\nabla}_\beta v^{\beta)}+v^\alpha\tilde{Q}_\alpha+cQ_4^*. \tag{3.4.26}$$

The terms proportional to c^{-2} that enter these equations describe internal relativistic effects, not connected with the motion of the continuous medium in relation to the observer with velocities close to that of light, and can be considered in the framework of Newtonian mechanics. If these terms can be neglected (which depends on the particular problem), Eqs. (3.4.25) and (3.4.26) coincide with the common equations for momentum and energy that allow for irreversible processes of viscosity and heat conduction.

3.5 CONCLUSION

The various macroscopic effects of special relativity, which disappear as $c \to \infty$, can be divided into two types.

First, there are the relativistic effects observed in the proper reference frame, with respect to which a point in the continuous medium is at rest at the given moment of time. Such effects do not exist for dust, for example. Second, there are the relativistic effects of special relativity due to the transition from the proper reference frame at each point of the medium to the observer's reference frame, which is the same for all points. These effects are associated with navigational calculations. We will call the effects of the first type, which have a thermodynamic nature, proper effects, and the effects of the second type, connected with employing the Lorentz transofrmation instead of Galilean, navigational effects. Proper relativistic effects, just as quantum efffects in some cases (e.g. ferromagnetism), manifest themselves only through their dependence on the parameters of the Lagrangian and the functional δW^* in proper reference frames and can be described in the framework of Newtonian mechanics. The equations of Newtonian mechanics that allow only for proper relativistic effects in local, proper reference frames coincide with the equations of relativistic mechanics, also written for proper reference frames.

The corresponding global equations of Newtonian mechanics can be obtained from the equations in proper reference frames by navigational conversion to global inertial reference frames associated with the observer via the Galilean transformation formulas, while to obtain relativistic equations that allow for all the relativistic effects we must use the Lorentz transformation. The equations of Newtonian mechanics derived in the above-discussed manner differ from the usual nonrelativistic equations for corresponding models of continuous media by terms proportional to c^{-2}. These additional terms generally have different orders depending on the order of the acceleration and other quantities of a thermodynamic nature (see Eqs. (3.3.2), (3.3.3), (3.4.25), and (3.4.26)).

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4

Models of Ferromagnetic Continuous Media with Magnetic Hysteresis

L. T. Chernyi, *Cand.Sc. (Physics and Mathematics)*
Institute of Mechanics, M. V. Lomonosov Moscow State University

INTRODUCTION

At present magnetic materials (magnetics) whose magnetization is nonzero are finding an ever growing field of applications [1, 2]. Experience shows that if we place a magnetic in an external magnetic field, its magnetization and stressed state generally change, and so does the magnetic field. Among the magnetics that are most strongly magnetizable and interact with the magnetic field are the ferromagnets; of the chemical elements at room temperature only iron, cobalt, nickel, and gadolinium are ferromagnets, while at very low temperatures terbium, dysprosium, holmium, erbium, and thulium are added. However, the number of ferromagnets is very great since they include alloys of the above nine elements and also alloys of any of these elements and a nonferrous element. Besides, there are ferromagnets that are alloys of nonferrous elements. For each ferromagnet there exists a certain critical temperature T_c known as the Curie point. Above this temperature the ferromagnet exhibits none of its specifically ferromagnetic properties. For the most common ferromagnets the Curie point lies between 10^2 and 10^3 degrees Celsius, e.g. for iron $T_c = 768$ °C, for cobalt $T_c = 1120$ °C, and for nickel $T_c = 358$ °C. In considering ferromagnetic media we will keep to the temperature range lying below the Curie point and will not study phenomena occurring at temperatures near the Curie point.

The intensity of magnetization at each point of the magnetic is characterized by the vector of the magnetic moment per unit volume of the substance, or simply magnetization, **M**. An important property of all ferromagnets is the existence of magnetic saturation, which means that the modulus M of the magnetization vector after attaining a certain value M_0 remains practically the same under a further increase in the magnetic field strength H. It is essential that ferromagnets exhibit no one-to-one dependence between **M** and **H** (for constant strains and temperatures), since for a given value of **H** we may have different values of **M** ($M < M_0$) at a given point depending on the previous magnetic history of this point, i.e. the magnetic field at that point in the past. This feature is well confirmed by the magnetic hysteresis of ferromagnets and the possibility of residual magnetization $\mathbf{M}^* \neq 0$ and $\mathbf{H} = 0$. A ferromagnetic body not magnetized to saturation consists of tiny regions of spontaneous magnetization known as ferromagnetic domains. In each domain the magnetization is equal to M_0, the saturation magnetization. But the individual magnetization vectors in various domains point in different directions, as a result of which the net magnetization may have a smaller (or even zero) value. The spontaneous magnetization of a domain depends on the temperature, and as this grows, M_0 diminishes and vanishes at the Curie point. Ferromagnetic domains have linear dimensions of about 10^{-7}-10^{-5} m, which is no greater in order of magnitude, than the size of crystallites (i.e. individual crystals) in a polycrystalline ferromagnet. Here we consider only cases where the characteristic linear dimensions in a specific problem are much greater than the sizes of crystallites and ferromagnetic domains. This enables us to consider polycrystalline ferromagnetic bodies as continuous media and assume that in a physically small region there are still many crystallites and domains.

The process of magnetization or remagnetization does not always proceed smoothly. It may occur by a succession of small jumps in the magnetization (the Barkhausen effect). This is caused by sudden remagnetization of some ferromagnetic domains in the substance or their parts. Usually this unevenness is not seen on the magnetization curve, since each jump is small and their number is great. The average volume remagnetized in each jump is, in order of magnitude, equal to the average volume of several domains. From the viewpoint of continuous media, we can take the change in magnetization in each jump to be infinitely small since by assumption in a physically small volume of a medium there is still a large number of domains. For this reason the process of magnetization of a ferromagnet in an infinitely slowly varying magnetic field is considered as a smooth and infinitely slow process of magnetization variation. As the reader must know, a cyclic remagnetization of a ferromagnet is accompanied by irreversible heat liberation even if the variations of $\mathbf{H}$ and $\mathbf{M}$ are infinitely small, provided that over a cycle the maximal value of $\mathbf{H}$ exceeds a certain value χ and magnetic hysteresis is present. Consequently, such a magnetization process can be considered from the macroscopical viewpoint as an irreversible process that proceeds infinitely slowly. Note that plastic deformation of a ferromagnet also constitutes an irreversible process, which may proceed arbitrarily slowly [1].

In an external magnetic field ferromagnets not only magnetize and become a source of magnetic field but may deform. The size of such magnetostrictive deformations, which appear in a free ferromagnetic body, may differ considerably depending on ferromagnetic material, temperature, magnetic field strength, and other factors and can be as high as 10^{-4}, while for rare-earth ferromagnets in very strong magnetic fields they can be as high as 10^{-3}. For solids this is quite a large deformation. Ferromagnets may also exhibit a phenomenon that is the converse of magnetostriction —a change in a ferromagnetic body's magnetization when the body is under stress. A nonuniformly magnetized ferromagnetic body or one whose surface is fixed by this or that means cannot, generally speaking, deform when magnetized so as to be free from internal stresses. For this reason, considerable stresses may appear in a body under magnetization. Internal stresses and plastic deformations in ferromagnetic bodies also influence the magnetization process.

A great number of studies have been devoted to ferromagnets, both because of purely scientific interest and because of the important practical applications that ferromagnetic materials find in science and technology. A great body of experimental data, the microscopic theory of magnetization, magnetic hysteresis, magnetostriction, and other properties of ferromagnets, and a large bibliography are contained in [2-5]. The interrelationship between the magnetic and mechanical properties of ferromagnets becomes increasingly important when magnetostriction is considerable since in this case the magnetic fields may cause considerable internal stresses, and the deformations may lead to variations in magnetization. Phenomenological models of magnetizable bodies (without hysteresis) allowing for both mechanical and magnetic properties are studied in [5-11]. The present article is devoted to building macroscopic models of ferromagnetic media with magnetic hysteresis. An essentially new feature here is the fact that the residual magnetization is taken as an additional determining parameters, for which we develop a kinetic equation. We also evolve the phenomenological theory of magnetic hysteresis and gyromagnetic phenomena. Irreversible magnetostriction and the influence of internal stresses on the magnetization process are also studied for bodies with residual magnetization and plastic deformations. The main relationships are derived from the basic variational equation of the mechanics of continuous media [12-14].

4.1 THE DETERMINING PARAMETERS

Let us consider within the framework of Newtonian mechanics the motion of a continuous medium in relation to the observer's reference frame with spatial coordinates x^i, a temporal coordinate t, basis ∂_i, and components g_{ij} of the metric tensor [1]. We also introduce a reference frame comoving with the medium and having Lagrangian coordinates ξ^α, temporal coordinate t, and basis ∂_α [2]. The relation between these two reference frames is given by the law of motion [1]

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[1] Latin indices can take on any value 1, 2, or 3 and are used to label components of tensors in the observer's reference frame.

[2] Greek indices can take on any value 1, 2, or 3 and are used to label components of tensors in the comoving reference frame.

$$x^i = x^i(\xi^\alpha, t).$$

The basis vector Э_α are related to the vectors Э_i via an affine transformation (at each point):

$$\text{Э}_\alpha = x^i{}_\alpha \text{Э}_i, \quad x^i{}_\alpha = \partial x^i/\partial\xi^\alpha; \quad \text{Э}_i = \xi_i{}^\alpha \text{Э}_\alpha, \quad \xi_i{}^\alpha = \partial\xi^\alpha/\partial x_i,$$

and the components of the metric tensor in the comoving system of coordinates are given by the formulas:

$$g_{\alpha\beta} = g_{ij} x^i{}_\alpha x^j{}_\beta.$$

We denote the contravariant components of the vectors of unit mass magnetization and magnetic induction by μ^γ and B^γ, respectively. We know that B^γ are represented thus:

$$B^\gamma = \varepsilon^{\gamma\alpha\beta}\nabla_\alpha A_\beta,$$

where $\varepsilon^{\gamma\alpha\beta}$ are the components of the Levi-Civita pseudotensor, ∇_α the covariant derivative, and A_β the components of the vector potential. We define the components of the magnetic field strength thus:

$$H^\gamma = B^\gamma - 4\pi\rho\mu^\gamma;$$

here ρ is the medium's density, with $\rho = \overset{\circ}{\rho}\,(|\overset{\circ}{g}_{\alpha\beta}|/|g_{\alpha\beta}|)^{\frac{1}{2}}$, where $\overset{\circ}{\rho}$ and $\overset{\circ}{g}_{\alpha\beta}$ are the medium's density and the components of the metric tensor in the initial state.

To allow for plastic deformations and residual magnetization we introduce the parameters $g^*_{\alpha\beta}$ and $\mu^{*\alpha}$. These acquire physical meaning if we visualize the following unloading process. We take an infinitely small particle with Lagrangian coordinates ξ^α, isolate it from the body, and in it reduce the magnetic field strength and internal stress tensor to zero. In the process the particle is deformed, the basis vector Э_α transform to Э^*_α, and the components $g_{\alpha\beta}$ of the metric tensor become $g^*_{\alpha\beta}$. The magnetization vector $\mu^\alpha \text{Э}_\alpha$ also changes and becomes $\mu^{*\alpha}\text{Э}^*_\alpha$. Next let us assume that the components $g^*_{\alpha\beta}$ and $\mu^{*\alpha}$ depend only on coordinates ξ^α and time t. The manifold with coordinates ξ^α, metric tensor components $g^*_{\alpha\beta}$, and contravariant components $\mu^{*\alpha}$ of the magnetization vector is the natural generalization of the idea of the state of a medium without internal stresses, introduced in plasticity theory [15, 16]. In the manifold we may introduce connectedness and by this transform it into an affinely connected manifold. Note that the curvature and torsion in this manifold are not zero, generally speaking, and the magnetic induction field is not solenoidal. All this points to the fact that it is impossible to ensure complete unloading of a particle of finite size.

We can associate the following physical parameters with $g^*_{\alpha\rho}$ and $\mu^{*\alpha}$:

(i) the reversible magnetization vector

$$\mu^\alpha_{(e)} = \mu^\alpha - \mu^{*\alpha},$$

(ii) the reversible strain tensor

$$\varepsilon^{(e)}_{\alpha\beta} = (1/2)(g_{\alpha\beta} - g^*_{\alpha\beta}),$$

and (iii) the residual strain tensor

$$\varepsilon^*_{\alpha\beta} = (1/2)\,(g^*_{\alpha\beta} - \overset{\circ}{g}_{\alpha\beta})$$

(here $\overset{\circ}{g}_{\alpha\beta}$ are the metric tensor components in the initial state).

Besides magnetization and magnetic field, we must allow for electric polarization and electric field, generally speaking. But in strong magnetics the ratio of the electric field to the magnetic field and the ratio of the dielectric susceptibility to the magnetic susceptibility are usually small (less than 10^{-5}). In what follows we will restrict ourselves to the quasi-stationary electromagnetic field approximation, as is usually done in magnetohydrodynamics [17]. This enables us to assume that polarization and electric field strength do not enter the set of determining parameters, while the magnetization, electric current, induction, and magnetic field strength are invariant under Galilean transformations [17].

If in addition we ignore chemical and phase transformations in magnetics and highly nonequilibrium processes in VHF fields (about 10^9 Hz) (see [2, 3, 5]), then to build the model of a magneto-elasto-plastic body that allows for magnetic hysteresis it suffices to take the following set of determining parameters:

$$v_\alpha,\ g_{\alpha\beta},\ \omega^\gamma,\ \mu^\alpha,\ \nabla_\beta\mu^\alpha,\ B^\gamma,\ g^*_{\alpha\beta},\ \mu^{*\alpha},\ S,\ K^{\nu_1\ldots\nu_m}_{(B)\lambda_1\ldots\lambda_n}. \qquad (4.1.1)$$

Here v_α are the linear velocity components of points in the medium, $\omega^\gamma = (1/2)\varepsilon^{\gamma\alpha\beta}\nabla_\alpha v_\beta$ the angular velocity components (introduced as a determining parameter to allow for gyromagnetic phenomena), S the entropy of unit mass of the medium, and $K^{\nu_1\ldots\nu_m}_{(B)\lambda_1\ldots\lambda_n}$ the set of tensor components that characterize the physical or geometric properties of the medium in the initial state ($\overset{\circ}{g}_{\alpha\beta}$ also belongs to $K^{\nu_1\ldots\nu_m}_{(B)\lambda_1\ldots\lambda_n}$). We assume, by definition, that $(dK^{\nu_1\ldots\nu_m}_{(B)\lambda_1\ldots\lambda_n}/dt)_{\xi^\alpha=\mathrm{const}} = 0$

4.2 THE VARIATIONAL PRINCIPLE AND THE MAIN EQUATIONS

In building models of continuous media it is expedient to start with the variational equation [12-14]

$$\delta \int_{t_1}^{t_2} \int_V \Lambda \rho \, dV \, dt + \delta W^* + \delta W = 0. \tag{4.2.1}$$

Here Λ is the Lagrangian depending on the parameters in (4.1.1), V an arbitrary region that envelops the particle of the medium we are considering, and dV the volume element:

$$dV = \sqrt{|g_{\alpha\beta}|} \, d\xi^1 \, d\xi^2 \, d\xi^3 = \sqrt{|g_{ij}|} \, dx^1 \, dx^2 \, dx^3.$$

The functional δW^*, which is introduced to allow for certain external influences and irreversible processes, is assumed given, while the functional δW is found from (4.2.1) and is represented by an integral along the boundary of a four-dimensional region $\{V, t\}$, with δW being linear in the variations of the determining parameters.

The determining parameters (4.1.1), on which Λ depends, may be expressed in terms of functions $x^k (\xi^\gamma, t)$, $\mu^\alpha (\xi^\gamma, t)$, $A_\alpha (\xi^\gamma, t)$, $S (\xi^\gamma, t)$, $g^*_{\alpha\beta} (\xi^\gamma, t)$, and $\mu^{*\alpha} (\xi^\gamma, t)$ and their derivatives with respect to position and time. The variations of these functions in (4.2.1) are assumed independent and are defined thus:

$$\delta x^k = x'^k (\xi^\gamma, t) - x^k (\xi^\gamma, t), \quad \delta\mu^\alpha = \mu'^\alpha (\xi^\gamma, t) - \mu^\alpha (\xi^\gamma, t),$$

$$\delta A_\alpha = A'_\alpha (\xi^\gamma, t) - A_\alpha (\xi^\gamma, t),$$

$$\delta g^*_{\alpha\beta} = g'^*_{\alpha\beta} (\xi^\gamma, t) - g^*_{\alpha\beta} (\xi^\gamma, t), \quad \delta\mu^{*\alpha} = \mu'^{*\alpha} (\xi^\gamma, t) - \mu^{*\alpha} (\xi^\gamma, t),$$

$$\delta S = S' (\xi^\gamma, t) - S (\xi^\gamma, t).$$

Then for δv_α, $\delta g_{\alpha\beta}$, $\delta\omega^\gamma$, $\delta\nabla_\beta\mu^\alpha$, and δB^γ we have the following formulas:

$$\delta v_\alpha = v^i x^r{}_\alpha \nabla_r \delta x_i + x^i{}_\alpha D \delta x_i, \quad \delta g_{\alpha\beta} = (x^r{}_\alpha x^j{}_\beta + x^r{}_\beta x^j{}_\alpha) \nabla_r \delta x_j,$$

$$\delta\omega^\gamma = (1/2) \varepsilon^{\gamma\alpha\beta} d (\nabla_\alpha \delta x_\beta)/dt + (\varepsilon^{\gamma\alpha\beta} \nabla_\alpha v^\nu - \omega^\gamma g^{\beta\nu}) \nabla_\beta \delta x_\nu,$$

$$\delta\nabla_\beta\mu^\alpha = \nabla_\beta \delta\mu^\alpha + \mu^\gamma \nabla_\beta \nabla_\gamma \delta x^\alpha, \quad \delta B^\gamma = \varepsilon^{\gamma\alpha\beta} \nabla_\alpha \delta A_\beta - B^\gamma \nabla_\alpha \delta x^\alpha,$$

where $D \equiv \partial/\partial t + v^k \nabla_k$, $\delta x^\alpha = \xi_k{}^\alpha \delta x^k$, $\delta x_\alpha = x^k{}_\alpha \, \delta x_k$, and $\delta x_k = g_{kr} \delta x^r$.

The components of the tensors $K_{(B)}$ are assumed to be known functions of ξ^α and are thereby not varied. Noting that $\delta (\rho \, dV) = 0$, for the first term in (4.2.1) we find

$$\begin{aligned}
&\delta \int\limits_t \int\limits_V \Lambda\rho\, dV\, dt = \int\limits_t \int\limits_V [\rho\,(\partial\Lambda/\partial S)\,\delta S + \rho\,(\delta\Lambda/\delta\mu^\alpha)\,\delta\mu^\alpha \\
&+ (\varepsilon^{\alpha\beta\gamma}\nabla_\beta\rho\,\partial\Lambda/\partial B^\gamma)\,\delta A_\alpha + \rho\,(\partial\Lambda/\partial\mu^{*\alpha})\,\delta\mu^{*\alpha} + \rho\,(\partial\Lambda/\partial g^*_{\alpha\beta})\,\delta g^*_{\alpha\beta} \\
&+ (\nabla_r T_k{}^r - \rho D p_k)\,\delta x^k]\, dV\, dt + \int\limits_t \int\limits_\Sigma \{\rho\,[\partial\Lambda/\partial\,(\nabla_\beta\mu^\alpha)]\,(\delta\mu^\alpha + \mu^\gamma\nabla_\gamma\delta x^\alpha) \\
&+ \varepsilon^{\gamma\beta\alpha}\rho\,(\partial\Lambda/\partial B^\gamma)\,\delta A_\alpha - T_\alpha{}^\beta\delta x^\alpha\}\, n_\beta\, d\Sigma\, dt \\
&+ \Big[\int\limits_V (\rho p^\alpha - (1/2)\,\varepsilon^{\gamma\beta\alpha}\nabla_\beta\rho k_\gamma)\,\delta x_\alpha\, dV + \int\limits_\Sigma (1/2)\,\varepsilon^{\gamma\beta\alpha}\rho k_\gamma\delta x_\alpha n_\beta\, d\Sigma\Big]_{t_1}^{t_2},
\end{aligned} \tag{4.2.2}$$

Here n_β are the components of a unit vector normal to the surface Σ that confine volume V, and we have introduced the following notations:

$$\begin{aligned}
&T^{\alpha\beta} = \nabla_\gamma\{\rho\,[\partial\Lambda/\partial\,(\nabla_\gamma\mu^\lambda)]\, g^{\lambda\alpha}\mu^\beta\} + (1/2)\,\rho d\,(\varepsilon^{\gamma\beta\alpha}k_\gamma)/dt \\
&+ \rho\varepsilon^{\gamma\beta\lambda}k_\gamma\nabla_\lambda v^\alpha + \rho k_\gamma\omega^\gamma g^{\alpha\beta} + \rho\,(\partial\Lambda/\partial B^\gamma)\, B^\gamma g^{\alpha\beta} - 2\rho\,(\partial\Lambda/\partial g_{\alpha\beta}) - \rho v^\alpha p^\beta, \\
&d/dt = (\partial/\partial t)_{\xi^\alpha = \mathrm{const}},\ k_\gamma = \partial\Lambda/\partial\omega^\gamma, \quad \delta\Lambda/\delta\mu^\alpha \\
&= \partial\Lambda/\partial\mu^\alpha - (1/\rho)\,\nabla_\gamma\rho\,(\partial\Lambda/\partial\nabla_\gamma\mu^\alpha), \quad p^\alpha = \partial\Lambda/\partial v_\alpha.
\end{aligned} \tag{4.2.3}$$

Let us define the functional δW^* thus:

$$\begin{aligned}
\delta W^* = &\int\limits_t \int\limits_V (\rho T\delta S + (1/c)\, j^\alpha\delta A_\alpha - \rho h_\alpha\delta\mu^\alpha - \rho R_\alpha\delta\mu^{*\alpha} \\
&- (1/2)\,\rho R^{\alpha\beta}\delta g^*_{\alpha\beta} - \tau_k{}^r\nabla_r\delta x^k)\, dV\, dt,
\end{aligned} \tag{4.2.4}$$

where T is the absolute temperature, j^α the components of the electric current vector, $\tau_k{}^r$ the components of the viscous stress tensor, and h_α, R_α, and $R^{\alpha\beta}$ are generalized forces determining the irreversible effects of magnetization relaxation, magnetic hysteresis, and plastic deformation (c is the speed of light in vacuum).

Assuming that the variations of variables on the surface of the four-dimensional region are zero (with $\delta W = 0$, by definition) and combining the variational equation (4.2.1) with (4.2.2)-(4.2.4), we arrive at the following equations:

$$-\partial\Lambda/\partial S = T, \quad \partial\Lambda/\partial\mu^\alpha - (1/\rho)\,\nabla_\gamma\rho\,(\partial\Lambda/\partial\nabla_\gamma\,\mu^\alpha) = h_\alpha, \tag{4.2.5}$$

$$-\varepsilon^{\beta\gamma\alpha}\,\nabla_\beta\rho\,(\partial\Lambda/\partial B^\gamma) = (1/c)\, j^\alpha, \quad \rho D p^k = \nabla_r\,(T^{kr} + \tau^{kr}), \tag{4.2.6}$$

$$\partial\Lambda/\partial\mu^{*\alpha} = R_\alpha, \quad 2\,(\partial\Lambda/\partial g^*_{\alpha\beta}) = R^{\alpha\beta}. \tag{4.2.7}$$

Equations (4.2.5) define temperature and generalize the equations of state for the reversible magnetization to the case of magnetization relaxation. Equations (4.2.6) are the Maxwell equations for the magnetic field and the momentum equations. Finally, Eqs. (4.2.7) serve to determine the residual magnetization and residual deformations (strains) and will be studied in Sec. 4.3.

If we introduce variations that are not zero on the boundary, we arrive at a formula for δW:

$$\delta W = -\int_t \int_\Sigma \{\rho\,[\partial\Lambda/\partial(\nabla_\beta\mu^\alpha)]\,(\delta\mu^\alpha + \mu^\gamma\nabla_\gamma\delta x^\alpha)$$

$$+\,\varepsilon^{\gamma\beta\alpha}\rho\,(\partial\Lambda/\partial B^\gamma)\,\delta A_\alpha - (T_\alpha{}^\beta + \tau_\alpha{}^\beta)\,\delta x^\alpha\}\,n_\beta\,d\Sigma\,dt$$

$$-\Big[\int_V [\rho p^\alpha - (1/2)\,\varepsilon^{\gamma\beta\alpha}\nabla_\beta\rho k_\gamma]\,\delta x_\alpha\,dV - \int_\Sigma (1/2)\,\varepsilon^{\gamma\beta\alpha}\rho k_\gamma\delta x_\alpha n_\beta\,d\Sigma\Big]_{t_1}^{t_2} \tag{4.2.8}$$

Fixing the value of δW on the boundary of the four-dimensional region $\{V, t\}$, we obtain the boundary conditions on Σ and the conditions at t_1 and t_2:

$$\begin{aligned}
&\rho\,[\partial\Lambda/\partial(\nabla_\beta\mu^\alpha)]\,n_\beta = M_\alpha,\quad \rho\varepsilon^{\gamma\beta\alpha}\,(\partial\Lambda/\partial B^\gamma)\,n_\beta = N^\alpha,\\
&(T_\alpha{}^\beta + \tau_\alpha{}^\beta)\,n_\beta = T_\alpha \quad \text{on} \quad \Sigma;\\
&\rho p^\alpha - (1/2)\,\varepsilon^{\gamma\beta\alpha}\nabla_\beta\rho k_\gamma = J_1^\alpha \quad \text{at} \quad t = t_1,\\
&\rho p^\alpha - (1/2)\,\varepsilon^{\gamma\beta\alpha}\nabla_\beta\rho k_\gamma = J_2^\alpha \quad \text{at} \quad t = t_2;\\
&(1/2)\,\varepsilon^{\gamma\beta\alpha}\rho k_\gamma n_\beta = Q_1^\alpha \quad \text{on} \quad \Sigma \quad \text{at} \quad t = t_1,\\
&(1/2)\,\varepsilon^{\gamma\beta\alpha}\rho k_\gamma n_\beta = Q_2^\alpha \quad \text{on} \quad \Sigma \quad \text{at} \quad t = t_2.
\end{aligned} \tag{4.2.9}$$

By means of the variational equation (4.2.1) we may also study the conditions at discontinuities [13, 14].

Equation (4.2.1) enables obtaining the energy equation. To this end we choose the independent variations in the following form:

$$\begin{aligned}
&\delta x^k = v^k\,\delta\varphi,\quad \delta\mu^\alpha = (d\mu^\alpha/dt)\,\delta\varphi,\quad \delta A_\alpha = (dA_\alpha/dt)\,\delta\varphi,\\
&\delta\mu^{*\alpha} = (d\mu^{*\alpha}/dt)\,\delta\varphi,\quad \delta g^*_{\alpha\beta} = (dg^*_{\alpha\beta}/dt)\,\delta\varphi,\quad \delta S = (dS/dt)\,\delta\varphi.
\end{aligned}$$

Here $\delta\varphi$ is an arbitrary constant, and d/dt stands for a time derivative with the ξ^α kept constant. Substituting these variations into Eq. (4.2.1) and taking into account the variation (4.2.8) and the fact that Eq. (4.2.1) is valid for any volume V and any t_1 and t_2, we obtain the energy equation

$$\rho\frac{d}{dt}(p^\alpha v_\alpha+k_\gamma\omega^\gamma-\Lambda)+\nabla_\beta\Big(\rho\frac{\partial\Lambda}{\partial\nabla_\beta\mu^\alpha}\Big[\frac{d\mu^\alpha}{dt}+\mu^\gamma\nabla_\gamma v^\alpha\Big]$$
$$+\varepsilon^{\gamma\beta\alpha}\rho\frac{\partial\Lambda}{\partial B^\gamma}\frac{dA_\alpha}{dt}-[T^{\alpha\beta}+\tau^{\alpha\beta}]\,v_\alpha\Big)-\rho T\frac{dS}{dt}$$
$$-\frac{1}{c}j^\alpha\frac{dA_\alpha}{dt}+\rho h_\alpha\frac{d\mu^\alpha}{dt}+\frac{1}{2}\rho R^{\alpha\beta}\frac{dg^*_{\alpha\beta}}{dt}+\rho R_\alpha\frac{d\mu^{*\alpha}}{dt}+\tau^{\alpha\beta}\nabla_\beta v_\alpha=0,$$

which is a corollary of Eqs. (4.2.3) and (4.2.5)-(4.2.7).

By employing the identity

$$(1/c)\,j^\alpha\,dA_\alpha/dt=-j^\alpha E'_\alpha-j^\alpha\nabla_\alpha\varphi'=-j^\alpha E'_\alpha$$
$$-\nabla_\alpha\,[c\varepsilon^{\alpha\beta\gamma}\rho\,(\partial\Lambda/\partial B^\gamma)\,\nabla_\beta\varphi'],$$

in which

$$E'_\alpha=x^k{}_\alpha\,[E_k+(1/c)\,\varepsilon_{kij}v^iB^j],\quad E_k=-(1/c)\,(\partial A_k/\partial t)-\nabla_k\varphi,$$
$$\varphi'=\varphi-(1/c)\,v^kA_k,$$

and E_k and φ are the electric field components and the electric potential in the observer's reference frame, we can transform the energy equation to the following form:

$$\rho\frac{d}{dt}(p^\alpha v_\alpha+k_\gamma\omega^\gamma-\Lambda)+\nabla_\beta\Big(\rho\frac{\partial\Lambda}{\partial\nabla_\beta\mu^\alpha}\Big[\frac{d\mu^\alpha}{dt}+\mu^\gamma\nabla_\gamma v^\alpha\Big]$$
$$-c\varepsilon^{\beta\alpha\gamma}E'_\alpha\rho\frac{\partial\Lambda}{\partial B^\gamma}-[T^{\alpha\beta}+\tau^{\alpha\beta}]\,v_\alpha\Big)-\rho T\frac{dS}{dt}$$
$$-j^\alpha E'_\alpha+\rho h_\alpha\frac{d\mu^\alpha}{dt}+\rho R_\alpha\frac{d\mu^{*\alpha}}{dt}+\rho R^{\alpha\beta}\frac{1}{2}\frac{dg^*_{\alpha\beta}}{dt}+\tau^{\alpha\beta}\nabla_\beta v_\alpha=0.$$

Now, to proceed with the building of models, we must introduce assumptions concerning the heat influx and the energy density of the continuous medium and the magnetic field. We assume that

$$\nabla_\alpha q^\alpha=-\rho T\,(dS/dt)+j^\alpha E'_\alpha+\rho h_\alpha\,(d\mu^\alpha/dt)$$
$$+(1/2)\,\rho R^{\alpha\beta}\,(dg^*_{\alpha\beta}/dt)+\rho R_\alpha\,(d\mu^{*\alpha}/dt)+\tau^{\alpha\beta}\nabla_\beta v_\alpha,\qquad(4.2.10)$$
$$\mathcal{E}=\rho\,(p^\alpha v_\alpha+k_\alpha\omega^\alpha-\Lambda),\qquad(4.2.11)$$

where q^α are the components of the heat flux vector, and $\mathcal{E}$ is the energy body density of the continuous medium and the magnetic field. Equation (4.2.10) can be thought of as the entropy balance equation, while Eq. (4.2.11) establishes a relationship between the total energy density and the Lagrangian.

The variational equation (4.2.1) can also be used to obtain equations that can be thought of as the equations for angular momentum. To this end we take the following independent variations:

$$\delta x^k = \varepsilon^k{}_{ij} r^i \delta\varphi^j, \quad \delta\mu^\alpha = \delta A_\alpha = \delta\mu^{*\alpha} = \delta g^*_{\alpha\beta} = \delta S = 0.$$

Here $r^i \mathfrak{Э}_i = \mathbf{r}\,(x^k)$ is the radius vector of a point with coordinates x^k in the observer's reference frame, and $\delta\varphi^j \mathfrak{Э}_j = \delta\varphi$ is an arbitrary constant vector. Substituting these variations into Eq. (4.2.1) and allowing for (4.2.4) and (4.2.8) and the fact that Eq. (4.2.1) is valid for any volume V and any values of t_1 and t_2, we obtain the equations

$$\rho D\,(\varepsilon_{ijk} r^j p^k + k_i) + \nabla_r\,(\rho\varepsilon_{ij}{}^k \mu^j\,[\partial\Lambda/\partial\,(\nabla_\gamma \mu^\alpha)]\,x^r_\gamma \xi^\alpha_k$$
$$- \varepsilon_{ijk} r^j\,[T^{kr} + \tau^{kr}]) - \varepsilon_{ijk}\tau^{jk} = 0, \qquad (4.2.12)$$

which can be considered as the equations for angular momentum and are a corollary of Eqs. (4.2.3) and (4.2.6). From (4.2.12) it follows that the vector with components k^γ can be taken as the intrinsic angular momentum density vector. It is usually assumed that in magnetics the intrinsic angular momentum is related to magnetization thus (see [2, 5]):

$$k^\gamma = \Gamma^\gamma_\alpha \mu^\alpha, \qquad (4.2.13)$$

where Γ^γ_α are the components of a tensor that belongs to the set $K_{(B)}$. Taking into account the equation of state $k_\gamma = \partial\Lambda/\partial\omega^\gamma$, we find that

$$\Lambda = \Gamma^\gamma_\alpha \mu^\alpha \omega_\gamma + \Lambda_0\,(v_\alpha, g_{\alpha\beta}, \mu^\alpha, \nabla_\alpha \mu^\gamma, B^\gamma, g^*_{\alpha\beta}, \mu^{*\alpha}, S). \qquad (4.2.14)$$

Next, by definition we may put

$$\Lambda_0 = (1/2)\,\iota^\alpha v_\alpha - [1/(8\pi\rho)]\,H^\alpha H_\alpha - u\,(g_{\alpha\beta}, \mu^\alpha, \nabla_\alpha \mu^\gamma, g^*_{\alpha\beta},$$
$$\mu^{*\alpha}, S, K_{(B)}). \qquad (4.2.15)$$

Using Eq. (4.2.11), we obtain the usual formula for the total energy density of the continuous medium and the magnetic field [5, 6]:

$$\mathscr{E} = (1/2)\,\rho v^\alpha v_\alpha + [1/(8\pi)]\,H^\alpha H_\alpha + \rho u, \qquad (4.2.16)$$

which clarifies the physical meaning of u and Λ. If instead of (4.2.15) we postulate (4.2.16) and that $p_\alpha = v_\alpha$, then we can obtain (4.2.15) from (4.2.11) and (4.2.14).

Equations (4.2.5), (4.2.6), (4.2.7) and (4.2.3) and the energy equation via substitution of the expression for Λ take the form

$$\partial u/\partial S = T, \quad \Gamma^{\gamma}_{\alpha}\omega_{\gamma} + H_{\alpha} - \partial u/\partial\mu^{\alpha} + (1/\rho)\, \nabla_{\gamma}\rho\, [\partial u/\partial(\nabla_{\gamma}\mu^{\alpha})]$$
$$= h_{\alpha}, \tag{4.2.17}$$

$$\operatorname{curl}^{\alpha} \mathbf{H} = (4\pi/c)\, j^{\alpha}, \quad \rho D v^{k} = \nabla_{r}\, (T^{kr} + \tau^{kr}), \tag{4.2.18}$$

$$-\partial u/\partial\mu^{*\alpha} = R_{\alpha}, \quad -2\, (\partial u/\partial g^{*}_{\alpha\beta}) = R^{\alpha\beta}, \tag{4.2.19}$$

$$T^{\alpha\beta} = 2\rho\, (\partial u/\partial g_{\alpha\beta}) - \nabla_{\gamma}\, [\rho\, \partial u/\partial\, (\nabla_{\gamma}\mu^{\lambda})\, g^{\lambda\alpha}\mu^{\beta}]$$
$$- \rho H^{\alpha}\mu^{\beta} + (1/2)\, \rho H_{\gamma}\mu^{\gamma} g^{\alpha\beta} + \rho k_{\gamma}\omega^{\gamma} g^{\alpha\beta}$$
$$+ \frac{1}{2}\rho \frac{d}{dt} \varepsilon^{\gamma\beta\alpha} k_{\gamma} + \rho\varepsilon^{\gamma\beta\lambda} k_{\gamma}\nabla_{\lambda} v^{\alpha} - \rho\, (k^{\alpha}\omega^{\beta} + k^{\beta}\omega^{\alpha})$$
$$+ \frac{1}{4\pi}\left(H^{\alpha}B^{\beta} - \frac{1}{2} H_{\gamma}B^{\gamma}g^{\alpha\beta}\right), \tag{4.2.20}$$

$$\rho \frac{d}{dt}\frac{\mathscr{E}}{\rho} + \nabla_{\gamma}\left(\frac{c}{4\pi}\varepsilon^{\gamma\alpha\beta}E'_{\alpha}H_{\beta} - \rho\frac{\partial u}{\partial\nabla_{\gamma}\mu^{\alpha}}\left[\frac{d\mu^{\alpha}}{dt} + \mu^{\lambda}\nabla_{\lambda}v^{\alpha}\right]\right.$$
$$\left. - [T^{\alpha\gamma} + \tau^{\alpha\gamma}]\, v_{\alpha} + q^{\gamma}\right) = 0. \tag{4.2.21}$$

To these we must add the following equations:

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{E} = -(1/c)\, (\partial\mathbf{B}/\partial t), \tag{4.2.22}$$

which are satisfied identically if $\mathbf{B}$ and $\mathbf{E}$ are expressed in terms of the potentials $\mathbf{A}$ and φ.

4.3 A PHENOMENOLOGICAL THEORY OF IRREVERSIBLE PROCESSES

We can transform the entropy balance equation (4.2.10) to the form

$$T\, [\rho\, dS/dt + \nabla_{\alpha}\, (1/T)\, q^{\alpha}] = \sigma_1 + \sigma_2, \tag{4.3.1}$$

where

$$\sigma_1 = \rho R_{\alpha}\, d\mu^{*\alpha}/dt + \rho R^{\alpha\beta}\, d\varepsilon^{*}_{\alpha\beta}/dt,$$
$$\sigma_2 = (1/T)\, q^{\alpha}\, \nabla_{\alpha} T + j^{\alpha}E'_{\alpha} + \tau^{\alpha\beta}\, \nabla_{\beta} v_{\alpha} + \rho h_{\alpha}\, d\mu^{\alpha}/dt.$$

Let us denote the entropy increment at the expense of internal irreversible processes by d_iS and assume that

$$\rho T\, d_iS = \sigma\, dt, \quad \sigma = \sigma_1 + \sigma_2. \tag{4.3.2}$$

The quantity σ_1 characterizes irreversible effects associated with plastic deformations and magnetic hysteresis, while σ_2 is associated with heat conduction, electric conduction, magnetization relaxation, and viscous dissipation. According to the second law of thermodynamics, the dissipation function σ is always nonnegative.

In the thermodynamics of irreversible processes one usually assumes that the dissipation function depends on the thermodynamic fluxes [1, 18, 19]. We will therefore assume that σ depends on the thermodynamic fluxes

$$d\mu^{*\alpha}/dt, \quad d\varepsilon^{*}_{\alpha\beta}/dt, \quad \nabla_\alpha T, \quad E'_\alpha, \quad \nabla_\beta v_\alpha, \quad d\mu^\alpha/dt,$$

the determining parameters (4.1.1), and some additional parameters χ_s that are given functionals of the collection of quantities (4.1.1).

We set the dependence of σ on its arguments and, following the general theory of irreversible processes [1, 18, 19], assume that the generalized forces

$$\rho R_\alpha, \quad \rho R^{\alpha\beta}, \quad -(1/T)\,q^\alpha, \quad j^\alpha, \quad \tau^{\alpha\beta}, \quad \rho h_\alpha$$

are given by the equalities

$$\begin{gathered}
\rho R_\alpha = \gamma_1 \frac{\partial\sigma}{\partial\,(d\mu^{*\alpha}/dt)}, \quad \rho R^{\alpha\beta} = \gamma_1 \frac{\partial\sigma}{\partial\,(d\varepsilon^{*}_{\alpha\beta}/dt)}, \\
-\frac{1}{T}\,q^\alpha = \gamma_2 \frac{\partial\sigma}{\partial\,(\nabla_\alpha T)}, \quad j^\alpha = \gamma_2 \frac{\partial\sigma}{\partial E'_\alpha}, \\
\tau^{\alpha\beta} = \gamma_2 \frac{\partial\sigma}{\partial\,(\nabla_\beta v_\alpha)}, \quad \rho h_\alpha = \gamma_2 \frac{\partial\sigma}{\partial\,(d\mu^\alpha/dt)},
\end{gathered} \tag{4.3.3}$$

which are simply a generalization of the Onsager principle. In (4.3.3) the derivatives are calculated with the parameters in (4.1.1) and χ_s kept constant, while the factors γ_1 and γ_2 are fixed so that the following equation is satisfied:

$$\sigma = \gamma_1\Gamma_1 + \gamma_2\Gamma_2,$$

where

$$\Gamma_1 = \frac{d\mu^{*\alpha}}{dt}\,\frac{\partial\sigma}{\partial\,(d\mu^{*\alpha}/dt)} + \frac{d\varepsilon^{*}_{\alpha\beta}}{dt}\,\frac{\partial\sigma}{\partial\,(d\varepsilon^{*}_{\alpha\beta}/dt)},$$

$$\Gamma_2 = \nabla_\alpha T\,\frac{\partial\sigma}{\partial\,(\nabla_\alpha T)} + E'_\alpha\,\frac{\partial\sigma}{\partial E'_\alpha} + \nabla_\beta v_\alpha\,\frac{\partial\sigma}{\partial\,(\nabla_\beta v_\alpha)} + \frac{d\mu^\alpha}{dt}\,\frac{\partial\sigma}{\partial\,(d\mu^\alpha/dt)}.$$

Let us study a class of models in which the thermodynamic fluxes $d\mu^{*\alpha}/dt$ and $d\varepsilon^{*}_{\alpha\beta}/dt$ enter only into σ_1, while $\nabla_\alpha T$, E'_α, $\nabla_\beta v_\alpha$, and $d\mu^\alpha/dt$ enter only into σ_2; both σ_1 and σ_2 are nonnegative. In this case the factors γ_1 and γ_2 are determined from the formulas $\sigma_1 = \gamma_1\Gamma_1$ and $\sigma_2 = \gamma_2\Gamma_2$.

As the main assumption for building models of plastic media with magnetic hysteresis we take the following idea: the entropy increment due to changes in residual deformations and residual magnetization depends solely on the increments of the two and not on the rates at which these increments occur. This assumption agrees well with the fact that even arbitrarily slow processes of plastic deformation and remagnetization in a body with magnetic hysteresis are irreversible. For σ_1 in this case we must take a homogeneous function of degree 1 ($\gamma_1 = 1$) in $d\mu^{*\alpha}/dt$ and $d\varepsilon^*_{\alpha\beta}/dt$. Then, combining (4.3.3) with (4.3.2), we find that

$$\rho R_\alpha = \frac{\partial \sigma_1}{\partial (d\mu^{*\alpha}/dt)}, \quad \rho R^{\alpha\beta} = \frac{\partial \sigma_1}{\partial (d\varepsilon^*_{\alpha\beta}/dt)}. \tag{4.3.4}$$

We know [1] that if the dissipation function is a homogeneous function of degree 1 with respect to thermodynamic fluxes, the generalized Onsager principle implies the existence of a loading function for the generalized forces and the associative law for the thermodynamic fluxes. Hence, the tensor components R_α and $R^{\alpha\beta}$, defined by (4.3.4), in the space of variables $\{R_\alpha, R^{\alpha\beta}\}$ lie on the surfaces

$$f_k (R_\alpha, R^{\alpha\beta}, \chi_s) = 0 \qquad (k = 1, \ldots, N), \tag{4.3.5}$$

which may be called loading surfaces, while for $d\mu^{*\alpha}/dt$ and $d\varepsilon^*_{\alpha\beta}/dt$ we have the associated law

$$d\mu^{*\alpha} = \sum_{k=1}^{N} d\lambda_k (\partial f_k/\partial R_\alpha), \quad d\varepsilon^*_{\alpha\beta} = \sum_{k=1}^{N} d\lambda_k (\partial f_k/\partial R^{\alpha\beta}). \tag{4.3.6}$$

Here $d\lambda_k$ are factors that should be determined with the help of (4.3.5) and the fact that σ_1 is nonnegative, with $d\lambda_k = 0$ at $f_k \neq 0$. We note also that the loading function f_k may depend on the determining parameters (4.1.1).

If the region G in the space of variables $\{R_\alpha, R^{\alpha\beta}\}$ is confined by the loading surfaces, is nonconcave, and contains the origin of coordinates, then from (4.3.6) follows the Mises maximum principle for magneto-plastic media:

$$R^{\alpha\beta}\, d\varepsilon^*_{\alpha\beta} + R_\alpha\, d\mu^{*\alpha} \geqslant R'^{\alpha\beta}\, d\varepsilon^*_{\alpha\beta} + R'_\alpha\, d\mu^{*\alpha},$$

where $R'^{\alpha\beta}$ and R'_α are any values of $R^{\alpha\beta}$ and R_α in G.

In what follows the functions f_k are chosen so that displacements in G correspond to the following inequalities

$$f_k = 0, \quad (\partial f_k/\partial R_\alpha)\, dR_\alpha + (\partial f_k/\partial^{\alpha\beta} R^{\alpha\beta})\, dR^{\alpha\beta} < 0.$$

Then $\sigma_1 \geqslant 0$ implies $d\lambda_k \geqslant 0$.

In considering irreversible effects related to heat conduction, electric conduction, magnetization relaxation, and viscous dissipation, we will assume [18] that σ_2 is a quadratic form ($\gamma_2 = 1/2$) in the thermodynamic fluxes $\nabla_\alpha T$, E'_α, $\nabla_\beta v_\alpha$, and $d\mu^\nu/dt$, with the coefficients of this form, generally speaking, depending on the determining parameters (4.1.1). Then (4.3.3) implies that the generalized forces $-q^\alpha/T$, j^α, $\tau^{\alpha\beta}$, and ρh_α are linear functions of the above-mentioned thermodynamic fluxes, and the matrix built on the coefficients of this linear function is symmetric.

But, as we also know [18], if there is a magnetic field and magnetization, the matrix built on the coefficients of the linear dependence of the generalized forces on the thermodynamic fluxes is generally not symmetric. This matrix may obviously be represented by the sum of a symmetric matrix coinciding with the matrix built on the coefficients of the quadratic form σ_2 and a skew-symmetric matrix that must be defined by an additional condition. Hence, Eqs. (4.3.3) for $-q^\alpha/T$, j^α, $\tau^{\alpha\beta}$, and ρh_α can be generalized in the following way:

$$\begin{aligned} -\frac{1}{T}q^\alpha &= \gamma_2\frac{\partial\sigma_2}{\partial(\nabla_\alpha T)}-\frac{1}{T}q'^\alpha, \quad j^\alpha=\gamma_2\frac{\partial\sigma_2}{\partial E'_\alpha}+j'^\alpha, \\ \tau^{\alpha\beta} &= \gamma_2\frac{\partial\sigma_2}{\partial(\nabla_\beta v_\alpha)}+\tau'^{\alpha\beta}, \quad \rho h_\alpha=\gamma_2\frac{\partial\sigma_2}{\partial(d\mu^\alpha/dt)}+\rho h'_\alpha. \end{aligned} \tag{4.3.7}$$

Here $-q'^\alpha/T$, j'^α, $\tau'^{\alpha\beta}$, and $\rho h'_\alpha$ are linear functions of the thermodynamic fluxes $\nabla_\alpha T$, E'_α, $\nabla_\beta v_\alpha$, and $d\mu^\alpha/dt$. The matrix of the coefficients of this linear law is skew-symmetric and for this reason

$$-(1/T)\,q'^\alpha\,\nabla_\alpha T+j'^\alpha E'_\alpha+\tau'^{\alpha\beta}\,\nabla_\beta v_\alpha+\rho h'_\alpha\,d\mu^\alpha/dt=0,$$

which shows that the additional terms in (4.3.7) contribute nothing to the dissipation function σ_2.

Equations (4.3.7) enables us to obtain the Fourier heat equation, Ohm's law, the Navier-Stokes equation, and the magnetization relaxation equations and to account for various crossed phenomena and effects associated with the influence of the magnetic field on heat conduction, electric conduction, viscous dissipation, and magnetization relaxation. These include the thermomagnetic and galvanomagnetic effects, which reveal themselves in ferromagnets in a special way (see [3]), e.g. exhibit hysteresis effects. Such anomalous properties of strong magnetics are taken into account in (4.3.7), since the coefficients in the linear dependence of the generalized forces on thermodynamic fluxes may depend not only on the magnetic field strength but on the size of the residual magnetization, which is determined by the magnetic history of the body.

If we do not allow for crossed effects, then the following is true:

$$\begin{aligned} &\sigma_2=\varkappa^{\alpha\beta}\nabla_\alpha T\nabla_\beta T+\rho^{\alpha\beta}E'_\alpha E'_\beta+\lambda^{\alpha\beta\gamma\delta}\nabla_\alpha v_\beta\nabla_\gamma v_\delta+\gamma_{\alpha\beta}\,(d\mu^\alpha/dt)\,(d\mu^\beta/dt), \\ &\varkappa^{\alpha\beta}=\varkappa^{\beta\alpha}, \quad \rho^{\alpha\beta}=\rho^{\beta\alpha}, \quad \lambda^{\alpha\beta\gamma\delta}=\lambda^{\gamma\delta\alpha\beta}, \quad \gamma_{\alpha\beta}=\gamma_{\beta\alpha}, \\ &-(1/T)\,q'^\alpha=\varkappa'^{\alpha\beta}\nabla_\beta T, \quad j'^\alpha=\rho'^{\alpha\beta}E'_\beta, \quad \tau'^{\alpha\beta}=\lambda'^{\alpha\beta\gamma\delta}\nabla_\gamma v_\delta, \\ &\rho h'_\alpha=\gamma'_{\alpha\beta}\,d\mu^\beta/dt, \quad \varkappa'^{\alpha\beta}=-\varkappa'^{\beta\alpha}, \quad \rho'^{\alpha\beta}=-\rho'^{\beta\alpha}, \\ &\lambda'^{\alpha\beta\gamma\delta}=-\lambda'^{\gamma\delta\alpha\beta}, \quad \gamma'_{\alpha\beta}=-\gamma'_{\beta\alpha}. \end{aligned} \tag{4.3.8}$$

In this case the additional terms in Eqs. (4.3.7) allow for the Hall and Leduc-Righi effects and the effect of the magnetic field and magnetization on the viscous stress tensor and the magnetization relaxation process. Substituting (4.3.8) into (4.3.7), we get the Fourier heat equation, Ohm's law, the Navier-Stokes equation, and the magnetization relaxation equations:

$$-(1/T)\,q^{\alpha} = (\varkappa^{\alpha\beta} + \varkappa'^{\alpha\beta})\,\nabla_{\beta} T, \quad j^{\alpha} = (\rho^{\alpha\beta} + \rho'^{\alpha\beta})\,E'_{\beta},$$
$$\tau^{\alpha\beta} = (\lambda^{\alpha\beta\gamma\delta} + \lambda'^{\alpha\beta\gamma\delta})\,\nabla_{\gamma} v_{\delta}, \quad \rho h_{\alpha} = (\gamma_{\alpha\beta} + \gamma'_{\alpha\beta})\,d\mu^{\beta}/dt. \tag{4.3.9}$$

Let us assume that the matrix $\| q^{\gamma\alpha} \|$ is the inverse of the matrix $\| \gamma_{\alpha\beta} + \gamma'_{\alpha\beta} \|$ and $q^{\lambda\alpha} = \gamma\varepsilon^{\lambda\beta\alpha}\mu_{\beta} + r^{\lambda\alpha}$, with $r^{\lambda\alpha} = r^{\alpha\lambda}$. If we allow for (4.2.17), the magnetization relaxation equations take the form

$$d\mu^{\alpha}/dt = \gamma\varepsilon^{\alpha\beta\lambda}\mu_{\beta}\rho h_{\lambda} + r^{\alpha\lambda}\rho h_{\lambda},$$
$$h_{\lambda} = \Gamma^{\alpha}_{\lambda}\omega_{\alpha} + H_{\lambda} - \partial u/\partial\mu^{\lambda} + (1/\rho)\,\nabla_{\gamma}\rho\,\partial u/\partial(\nabla_{\gamma}\mu^{\lambda}). \tag{4.3.10}$$

These relationships can be considered as a generalization of the Landau equations (see [5]) to the case of an accelerated stressed medium. If we allow for crossed effects, we can write Eqs. (4.3.10) thus:

$$d\mu^{\alpha}/dt = (\gamma\varepsilon^{\alpha\beta\lambda}\mu_{\beta} + r^{\alpha\lambda})\,(\rho h_{\lambda} + a_{\lambda}^{\alpha\beta}\nabla_{\beta} v_{\alpha} + b_{\lambda}{}^{\beta}E'_{\beta} + c_{\lambda}{}^{\beta}\nabla_{\beta} T),$$

where γ, $r^{\alpha\lambda}$, $a_{\lambda}^{\alpha\beta}$, $b_{\lambda}{}^{\beta}$, and $c_{\lambda}{}^{\beta}$ are phenomenological coefficients that may depend on the determining parameters (4.1.1).

The above theory was developed for an accelerated anisotropic medium with finite strains. The relationships simplify if we assume that the medium is isotropic and the strains small.

4.4 SOME COROLLARIES OF THE GENERAL THEORY

4.4.1. Irreversible Magnetostriction. We know (e.g. see [3]) that when magnetized, magnetics deform, i.e. magnetostriction occurs. In strong magnetics, in addition, the residual deformations also change [3]. We will call the changes in $\varepsilon^{(e)}_{\alpha\beta}$ and $\varepsilon^{*}_{\alpha\beta}$ due to magnetization the reversible and irreversible magnetostrictions. The latter, as a rule, is much larger than the former and plays the main role. Hence, a change in $\mu^{*\alpha}$ is generally accompanied by a change in $\varepsilon^{*}_{\alpha\beta}$, and vice versa.

Let us study these phenomena by applying the theory we have just developed. Suppose that u does not depend on the magnetization gradients, while $\tau^{\alpha\beta} = 0$ and $h_{\alpha} = 0$. Then without loss of generality we can put $u = u\,(S, \mu^{*\gamma}, \varepsilon^{*}_{\alpha\beta}, \mu^{\gamma}_{(e)}, \varepsilon^{(e)}_{\alpha\beta}, K_{(B)})$. Using Eqs. (4.2.17), (4.2.19), and (4.2.20), the definitions for $\varepsilon^{(e)}_{\alpha\beta}$, $\varepsilon^{*}_{\alpha\beta}$, $\mu^{\gamma}_{(e)}$, and $\mu^{*\gamma}$, and neglecting gyromagnetic effects (which are usually small), we find that

$$H_\alpha = \partial u/\partial \mu^\alpha_{(e)}, \quad T^{\alpha\beta} = \rho p^{\alpha\beta} + (1/4\pi)\,[H^\alpha H^\beta - (1/2)\,H^\nu H_\gamma g^{\alpha\beta}],$$

$$R_\alpha = H_\alpha - \partial u/\partial \mu^{*\alpha}, \quad R^{\alpha\beta} = p^{\alpha\beta} - \partial u/\partial \varepsilon^*_{\alpha\beta},$$

where $p^{\alpha\beta} = \partial u/\partial \varepsilon^{(e)}_{\alpha\beta}$.

Instead of u we introduce the function $F = u - TS$ and assume that $F = F_0\,(T,\ \varepsilon^*_{\alpha\beta},\ \mu^{*\gamma}) + F_1\,(T,\ \varepsilon^{(e)}_{\alpha\beta},\ \mu^\gamma_{(e)})$, while the loading functions f_k depend only on $R^{\alpha\beta}$, R_γ, $\varepsilon^*_{\alpha\beta}$, $\mu^{*\gamma}$, T, χ_s, and $K_{(B)}$. In this case we have

$$R^{\alpha\beta} = p^{\alpha\beta} - \partial F_0/\partial \varepsilon^*_{\alpha\beta}, \quad p^{\alpha\beta} = \partial F_1/\partial \varepsilon^{(e)}_{\alpha\beta},$$

$$R_\gamma = H_\gamma - \partial F_0/\partial \mu^{*\gamma}, \quad H_\gamma = \partial F_1/\partial \mu^\gamma_{(e)}$$

and can put $f_k = f_k\,(p^{\alpha\beta},\ H_\gamma,\ \varepsilon^*_{\alpha\beta},\ \mu^{*\gamma},\ \chi_s,\ T)$. Then we can write the law (4.3.6) in the following form:

$$d\varepsilon^*_{\alpha\beta} = \sum_{k=1}^{N} d\lambda_k\,(\partial f_k/\partial p^{\alpha\beta}), \quad d\mu^{*\gamma} = \sum_{k=1}^{N} d\lambda_k\,(\partial f_k/\partial H_\gamma), \tag{4.4.1}$$

where $d\lambda_k \geqslant 0$ at $f_k = 0$ and $d\lambda_k = 0$ at $f_k \neq 0$.

Suppose that as $\varepsilon^*_{\alpha\beta}$ and $\mu^{*\gamma}$ change, $f_k = 0$ for only one value of k, i.e. $k = \nu$, and the increments $d\chi_s$ are linear functions of $d\varepsilon^*_{\alpha\beta}$ and $d\mu^{*\gamma}$ via the equalities $d\chi_s = A^{\alpha\beta}_{(s)} d\varepsilon^*_{\alpha\beta} + A_{(s)\gamma}\,d\mu^{*\gamma}$, where $A^{\alpha\beta}_{(s)}$ and $A_{(s)\gamma}$ are known functions of the determining parameters (4.1.1) and the χ_s. Then Eqs. (4.4.1) and the condition $df_\nu = 0$ yield that at $d\lambda_\nu > 0$

$$d\lambda_\nu = r_\nu\, d'f_\nu, \text{ where } d'f_\nu = (\partial f_\nu/\partial p^{\alpha\beta})\,dp^{\alpha\beta} + (\partial f_\nu/\partial H_\gamma)\,dH_\gamma + (\partial f_\nu/\partial T)\,dT, \tag{4.4.2}$$

with the factor r_ν expressed in terms of $A^{\alpha\beta}_{(s)}$ and $A_{(s)\gamma}$ and partial derivatives of f_ν and depending on the χ_s and the determining parameters.

Substituting the expressions for $d\lambda_\nu$ into Eqs. (4.4.1) and bearing in mind that $d\lambda_k = 0$ at $k \neq \nu$, we obtain

$$\begin{aligned} d\varepsilon^*_{\alpha\beta} &= r_\nu \frac{\partial f_\nu}{\partial p^{\alpha\beta}} \left(\frac{\partial f_\nu}{\partial p^{\nu\delta}}\,dp^{\nu\delta} + \frac{\partial f_\nu}{\partial H_\gamma}\,dH_\gamma + \frac{\partial f_\nu}{\partial T}\,dT \right), \\ d\mu^{*\gamma} &= r_\nu \frac{\partial f_\nu}{\partial H_\gamma} \left(\frac{\partial f_\nu}{\partial p^{\alpha\delta}}\,dp^{\alpha\delta} + \frac{\partial f_\nu}{\partial H_\alpha}\,dH_\alpha + \frac{\partial f_\nu}{\partial T}\,dT \right). \end{aligned} \tag{4.4.3}$$

This implies that when $\varepsilon^*_{\alpha\beta}$ and $\mu^{*\gamma}$ change with T kept constant and only one loading function is zero,

$$\left(\frac{d\varepsilon^*_{\alpha\beta}}{dH_\gamma} \right)_{p^{\delta\lambda},\ T=\text{const},\ d\lambda_\nu>0} = \left(\frac{d\mu^{*\gamma}}{dp^{\alpha\beta}} \right)_{H_\lambda,\ T=\text{const},\ d\lambda_\nu>0}, \tag{4.4.4}$$

where $d\varepsilon^*_{\alpha\beta}/dH_\gamma$ and $d\mu^{*\gamma}/dp^{\alpha\beta}$ stand for the ratios of infinitesimal increments of the respective quantities.

Equations (4.4.4) establish the relationship between irreversible magnetostriction and the variation in residual magnetization under internal stresses. These irreversible phenomena constitute crossed effects, which cannot be described within the scope of Onsager's linear theory. In deriving Eqs. (4.4.4) we used ideas of nonlinear thermodynamics of irreversible processes, which were first formulated in plasticity theory, but in fact have a broader thermodynamic nature [1, 19].

We also note that

$$p^{\alpha\beta} = \partial F_1/\partial\varepsilon^{(e)}_{\alpha\beta},\ H_\gamma = \partial F_1/\partial\mu^\gamma_{(e)},\ d\,(F_1 - p^{\alpha\beta}\varepsilon^{(e)}_{\alpha\beta} - H_\gamma\mu^\gamma_{(e)})_{T=\text{const}}$$
$$= -\varepsilon^{(e)}_{\alpha\beta}\,dp^{\alpha\beta} - \mu^\gamma_{(e)}\,dH_\gamma$$

imply that

$$\left(\frac{\partial\varepsilon^{(e)}_{\alpha\beta}}{\partial H_\gamma}\right)_{p^{\delta\lambda},\ T=\text{const}} = \left(\frac{\partial\mu^\gamma_{(e)}}{\partial p^{\alpha\beta}}\right)_{H_\lambda,\ T=\text{const}},$$

which establish the relationship between magnetostriction and variations in magnetization due to internal stresses in reversible processes [3].

Let us assume that (4.4.2) also hold for the case where several functions f_ν vanish when $\varepsilon^*_{\alpha\beta}$ and $\mu^{*\gamma}$ change. Then Eqs. (4.4.3) remain valid if we assume that in them we sum over all indices ν for which $f_\nu = 0$. Equations (4.4.4) are then valid for those increments $dp^{\alpha\beta}$ and dH_γ for which the factors $d\lambda_\nu$ with the same value of ν are positive on both sides of these equations. If $r_\nu > 0$, the following inequality follows from Eqs. (4.4.2) and (4.4.3) with T kept constant:

$$dp^{\alpha\beta}\,d\varepsilon^*_{\alpha\beta} + dH_\gamma\,d\mu^{*\gamma} = \sum_\nu r_\nu\,(d'f_\nu)^2 \geqslant 0,$$

which is a generalization of a similar inequality in plasticity theory.

Let us study the relationship between the residual deformations and common plastic deformations. Suppose that a fraction of the loading functions f_p depends on $R^{\alpha\beta}$ and R_γ only through the sum $N_\gamma = R_\gamma + 2\Gamma_{\alpha\beta\delta\gamma}R^{\alpha\beta}\mu^{*\delta}$, where the coefficients $\Gamma_{\alpha\beta\delta\gamma}$ are symmetric in δ and γ and belong to $K_{(B)}$, while the rest of the loading functions f_q do not depend on R_γ. Then the law (4.3.6) takes the form

$$d\mu^{*\gamma} = \sum_p d\lambda_p\,(\partial f_p/\partial N_\gamma),\ d\varepsilon^*_{\alpha\beta} = \sum_q d\lambda_q\,(\partial f_q/\partial R^{\alpha\beta}) + d\,(\Gamma_{\alpha\beta\delta\gamma}\mu^{*\delta}\mu^{*\gamma}).$$
$$(4.4.5)$$

In this case we may assume that the residual deformations consist of magnetostriction deformations due to residual magnetization and to common plastic deformations. But there is no way in which we can do a similar decomposition for an arbitrary dependence of the f_k on $R^{\alpha\beta}$ and R_γ.

4.4.2. Typical Magnetic Hysteresis Loops. In strong magnetics we can observe magnetic hysteresis phenomena in which the form of the hysteresis loop strongly depends on internal stresses [3]. Let us study these phenomena by applying the above theory. We take a process in which a long cylindrical sample of a magnetoelastic material is remagnetized and stretched or compressed by a uniform external magnetic field directed along the cylinder's axis and by external forces applied at its faces. Assuming the deformations to be small, we can write the equation for the increment of the residual magnetization in the form

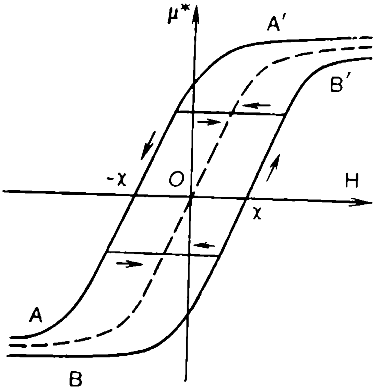

Fig. 4.1.

$$\begin{gathered} d\mu^* = d\lambda\,(\partial f/\partial H), \quad f = f(p,\ H,\ \varepsilon^*,\ \mu^*,\ \chi_s,\ T), \\ d\lambda \geqslant 0 \ \text{at}\ f = 0, \quad d\lambda = 0 \ \text{at}\ f \neq 0, \end{gathered} \tag{4.4.6}$$

where ε^* is the residual elongation along the cylinder's axis (the x^1 axis), p the component p^{11} of the normal stress on the cross-sectional area perpendicular to the cylinder's axis, μ^* the size of the residual magnetization directed along the x^1 axis, and H the magnetic field strength.

Let us assume that the loading function has the form

$$f = (H - \varphi(\mu^*))^2 - \chi^2. \tag{4.4.7}$$

Solving Eq. (4.4.6) together with (4.4.7), we obtain the magnetization diagram, which in Fig. 4.1 is depicted by solid curves. Curves

AA' and BB' reflect the dependence $H = \varphi(\mu^*) \pm \chi$, and the dashed curve the dependence $H = \varphi(\mu^*)$. The parameter χ in this case is the coercive force.

The remagnetization process occurs in this way only if the sample was previously magnetized to the saturated state [3]. If this condition is not met, the remagnetization diagram has a more complex appearance, since if the residual magnetization exceeds a certain critical value, χ may begin changing. But if the sample was previously magnetized up to saturation, this critical value is simply the saturation magnetization, and the coercive force remains constant. To formulate the law that governs the behavior of χ, let us consider a typical idealized magnetization diagram of a sample that prior to the experiment was demagnetized by heating above the Curie point and then cooling outside the magnetic field. The diagram is depicted in Fig. 4.2.

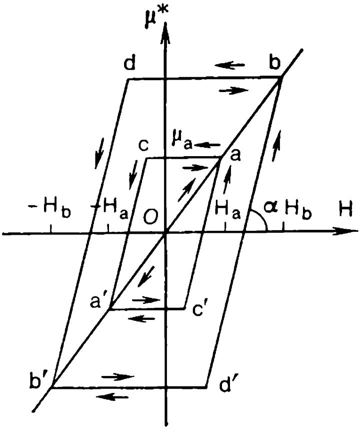

Fig. 4.2

The variation of μ^* under the action of a magnetic field whose absolute strength grows monotonically starting from zero is depicted by the initial magnetization curve (straight line $b'a'Oab$) with the equation $H = q\mu^*$. Suppose that the process of initial magnetization terminated at point a on the straight line $a'Oa$. Then, when subsequently the magnetic field strength changes to $-H_a$ and from $-H_a$ to $+H_a$, we obtain the remagnetization curve $aca'c'a$, which follows from the theory we have developed if the loading function has the form $f = (H - k\mu^*)^2 - \chi^2$, where $k = \cot \alpha$ and $\chi = (q - k)\mu_a^*$. But if we have reached point a or a' on the hysteresis loop $aca'c'a$ and the absolute value of the magnetic strength is again growing, the variation in μ^* is represented by a section of the initial magnetization curve emerging from point a or a'.

In the process the coercive force χ grows in such a way that the condition $f = 0$, the one condition for μ^* to change, is retained. When subsequently the magnetic field strength changes to $-H_b$ and from $-H_b$ to $+H_b$; the resulting remagnetization curve is $bdb'd'b$. Points a and a' are the points of intersection of the initial magnetization curve with straight lines ac' and $a'c$. Therefore, at these points $f(q\mu^*, \mu^*, \chi) = (q\mu^* - k\mu^*)^2 - \chi^2 = 0$. From the magnetization diagram shown in Fig. 4.2. It follows that this condition remains valid when χ changes.

Let us now formulate the law of variation of χ. Suppose that $f(H, \mu^*, \chi)$ is the loading function, and $H = \psi(\mu^*)$ is the equation of the initial magnetization curve. We assume, by definition, that

$$\begin{aligned} d\chi &= 0 \quad \text{at} \quad F \equiv f(\psi(\mu^*), \mu^*, \chi) \neq 0, \\ d\chi &\neq 0 \quad \text{at} \quad F \equiv f(\psi(\mu^*), \mu^*, \chi) = 0, \end{aligned} \tag{4.4.8}$$

if this is compatible with the conditions that $dF = 0$ and $d\lambda \geqslant 0$; otherwise $d\chi = 0$ when F is zero, as well. If $f = (H - k\mu^*)^2 - \chi^2$

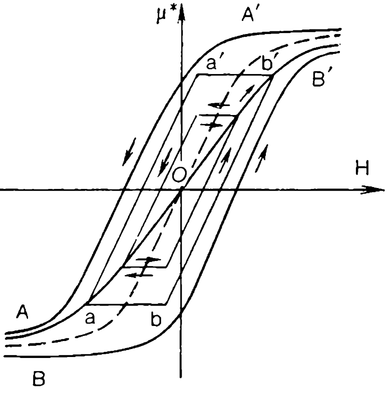

Fig. 4.3

and $\psi(\mu^*) = q\mu^*$ with $q > k$, from (4.4.6) and (4.4.8) there follows the magnetization diagram depicted in Fig. 4.2. At $k = 0$ the hysteresis loops in Fig. 4.2 have the shape of rectangles.

Solving Eqs. (4.4.6) and (4.4.8) for the case where the loading function is given by (4.4.7) and the initial magnetization curve by the equation $H = \varphi(\mu^*) + q\mu^*$, we arrive at a magnetization diagram depicted in Fig. 4.3. The lines aa', bb', AA', and BB' represent the function $H = \varphi(\mu^*) \pm \chi$ for different values of χ, and the dashed line and line aOb' the functions $H = \varphi(\mu^*)$ and $H = \varphi(\mu^*) + q\mu^*$, respectively.

Since the reversible magnetization $\mu_{(e)}$ depends on H, we can easily obtain diagrams representing the variation of the total magnetization under the action of a magnetic field. Obviously, we also obtain hysteresis loops in this case.

In the general three-dimensional case, for each loading function f_k we can introduce a parameter χ_k. If the loading function and the equations of the initial loading curve have the form

$$f_k = f_k (R^{\alpha\beta}, R_\gamma, \varepsilon^*_{\alpha\beta}, \mu^{*\gamma}, T, \chi_s), \quad R^{\alpha\beta} = \psi^{\alpha\beta} (\varepsilon^*_{\gamma\lambda}, \mu^{*\gamma}, T),$$

$$R_\gamma = \psi_\gamma (\varepsilon^*_{\alpha\lambda}, \mu^{*\gamma}, T),$$

then by definition we assume that

$$\begin{aligned} d\chi_k &= 0 \quad \text{at} \quad F_k (\varepsilon^*_{\alpha\beta}, \mu^{*\gamma}, T, \chi_s) \neq 0, \\ d\chi_k &\neq 0 \quad \text{at} \quad F_k (\varepsilon^*_{\alpha\beta}, \mu^{*\gamma}, T, \chi_s) = 0, \end{aligned} \tag{4.4.9}$$

if this is compatible with the conditions that $dF_k = 0$ and $d\lambda_r \geqslant 0$; otherwise $d\chi_k = 0$ when $F_k = 0$ as well. Here $F_k \equiv f_k (\psi_{\alpha\beta} (\varepsilon^*_{\delta\lambda}, \mu^{*\gamma}, T), \psi_\alpha (\varepsilon^*_{\delta\lambda}, \mu^{*\gamma}, T), \varepsilon^*_{\alpha\beta}, \mu^{*\gamma}, T, \chi_r)$. The parameters χ_k do not exhaust the entire collection of parameters χ_s. For instance, the χ_s may include χ_1 and χ_2 whose increments are given thus: $d\chi_1 = R_\alpha \, d\mu^{*\alpha}$ and $d\chi_2 = R^{\alpha\beta} \, d\varepsilon^*_{\alpha\beta}$.

Let us now study the effect of elastic stresses on the magnetization of the cylinder. We assume that the loading function is $f = f(H + 2\alpha\mu^* p, \mu^*, \varepsilon^*, \chi)$. Then from Eqs. (4.4.5) we find that $\varepsilon^* = \alpha\mu^{*2} + \text{const}$, while the condition $f = 0$ for the lateral sides of the hysteresis loop yields equations of the type

$$H = \varphi_{1,2} (\varepsilon^*, \mu^*, \chi) - 2\alpha p \mu^*.$$

This implies that in materials with positive magnetostriction ($\alpha > 0$) that is a quadratic function of the residual magnetization, elastic stretching ($p > 0$) must lead to a steeper hysteresis loop, and elastic compression ($p < 0$) to a flatter loop. In materials with negative magnetostriction ($\alpha < 0$) this pattern is reversed. This relationship between the sign of the magnetostriction effect and the influence of the type of elastic stresses on the shape of the hysteresis loop has been observed in experiment [3]. Plastic deformation also has a marked effect on the magnetization process [3]. For this reason it is highly important that the loading functions have among their arguments $\varepsilon^*_{\alpha\beta}$ that characterize irreversible magnetization.

4.4.3. Gyromagnetic Effects. Let us consider the main gyromagnetic phenomena. From Eqs. (4.2.17) it follows that in its influence on magnetization, rotation is equivalent to the action of an additional magnetic field of strength $\Gamma^\gamma{}_\alpha \omega_\gamma$. For this reason rotating a body outside a magnetic field leads to magnetization (the Barnett effect).

If $\tau^{ij} = \tau^{ji}$, Eqs. (4.2.12) and the boundary conditions (4.2.9) lead to the following relationship:

$$\frac{d}{dt}\int_V (\varepsilon^{\alpha\beta\gamma} r_\beta v_\gamma + k^\alpha)\, \text{Э}_\alpha \rho \, dV + \int_\Sigma \varepsilon^{\alpha\beta\gamma} (\mu_\beta M_\gamma - r_\beta T_\gamma)\, \text{Э}_\alpha \, d\Sigma = 0, \tag{4.4.10}$$

which can be taken as the angular momentum equation. Suppose that the boundary conditions (4.2.9) are such that the surface integral in (4.4.10) yields zero. Then

$$\int_V (\varepsilon^{\alpha\beta\gamma} r_\beta v_\gamma + k^\alpha)\, \text{Э}_\alpha \rho \, dV = \text{const.}$$

According to Eq. (4.2.13), $k^\gamma = \Gamma^\gamma{}_\alpha \mu^\alpha$. Hence, if we magnetize a body consisting of a ferromagnetic material in the absence of external torques or force couples, the body acquires a rotation (the Einstein-de Haas effect).

4.5 EXAMPLES OF MODELS OF MAGNETIZABLE MEDIA

4.5.1. The Model of Magnetoelastic Body. Here by definition we assume that

$$g^*_{\alpha\beta} = \mathring{g}_{\alpha\beta}(\xi^\gamma), \quad \varepsilon^*_{\alpha\beta} = 0, \quad \mu^{*\gamma} = 0, \quad \varepsilon_{\alpha\beta} = (1/2)(g_{\alpha\beta} - \mathring{g}_{\alpha\beta}),$$

$$\rho\Lambda = (1/2)\rho v^k v_k - (1/8\pi) H^k H_k - \rho u (S,\ \varepsilon_{\alpha\beta},\ \mu^\gamma,\ K_{(B)}),$$

$$\delta W^* = \int_t \int_V [\rho T \delta S + (1/c) j^\alpha \delta A_\alpha - \rho h_\alpha \delta\mu^\alpha - \tau_k{}^r \nabla_r \delta x^k]\, dV\, dt.$$

In this case $\sigma_1 = 0$ and the quantities $g^*_{\alpha\beta}$ are no longer unknown and belong to the set $K_{(B)}$. The complete set of equations consists of

$$\begin{aligned} &\partial u/\partial S = T, \quad H_\alpha - \partial u/\partial\mu^\alpha = h_\alpha, \\ &\rho D v^k = \nabla_r (T^{kr} + \tau^{kr}), \quad \operatorname{curl}^k \mathbf{H} = (4\pi/c) j^k, \\ &\operatorname{curl}^k \mathbf{E} = -(1/c)\, \partial B^k/\partial t, \quad \operatorname{div} \mathbf{B} = 0, \end{aligned} \tag{4.5.1}$$

the kinetic equations (4.3.9), and the energy equation (4.2.21), instead of which we may take the entropy balance equation (4.3.1). According to (4.2.20),

$$T^{\alpha\beta} = \rho\, \partial u/\partial\varepsilon_{\alpha\beta} + (1/4\pi)[H^\alpha H^\beta - (1/2) H^\gamma H_\gamma g^{\alpha\beta}]. \tag{4.5.2}$$

If instead of u we introduce the function $u' = u + 2\pi\rho\mu^\gamma\mu_\gamma$, Eqs. (4.2.16) for the total energy density $\mathscr{E}$ and (4.5.2) for the components $T^{\alpha\beta}$ can be transformed into

$$\mathscr{E} = (1/2)\rho v^{\gamma} v_{\gamma} + \rho u' + (1/8\pi) H_{\alpha} B^{\alpha} - (1/2)\rho\mu_{\alpha} B^{\alpha},$$

$$T^{\alpha\beta} = [\rho\,(\partial u'/\partial\varepsilon_{\alpha\beta}) + (1/2)\,\rho\mu_{\gamma} B^{\gamma} g^{\alpha\beta} - \rho B^{\alpha}\mu^{\beta}]$$

$$+ (1/4\pi)\,[H^{\alpha} B^{\beta} - (1/2)\, H^{\gamma} B_{\gamma} g^{\alpha\beta}].$$

These formulas coincide with the expressions for $\mathscr{E}$ and $T^{\alpha\beta}$ obtained by Sedov [6].

When $u = u\,(S, \rho, \mu^{\alpha}, K_{(B)})$, the model can be taken as a generalization of a model used in magnetohydrodynamics that allows for the fluid's magnetization.

4.5.2. The Model of a Magnetoelastic Body with Plastic Deformations and Magnetic Hysteresis. Here

$$\rho\Lambda = (1/2)\,\rho v^{\gamma} v_{\gamma} - (1/8\pi)\, H^{\gamma} H_{\gamma} - \rho u\,(S,\ \varepsilon_{\alpha\beta},\ \varepsilon^{*}_{\alpha\beta},\ \mu^{\gamma},\ \mu^{*\gamma},\ K_{(B)}),$$

$$\varepsilon_{\alpha\beta} = (1/2)\ (g_{\alpha\beta} - \overset{\circ}{g}_{\alpha\beta}),$$

$$\delta W^{*} = \int\limits_{t}\int\limits_{V} [\rho T\delta S + (1/c)\, j^{\alpha}\delta A_{\alpha} - \rho h_{\alpha}\delta^{\alpha}_{\mu}$$

$$- \tau_{k}{}^{r}\nabla_{r}\delta x^{k} - \rho R_{\alpha}\delta\mu^{*\alpha} - \rho R^{\alpha\beta}\delta\varepsilon^{*}_{\alpha\beta}]\, dV\, dt.$$

The complete set of equations consists of Eqs. (4.5.1), (4.5.2), the kinetic equations (5.3.9), the energy equation (4.2.21) or the entropy balance equation (4.3.1), and the following equation for determining $\varepsilon^{*}_{\alpha\beta}$ and $\mu^{*\gamma}$:

$$d\varepsilon^{*}_{\alpha\beta} = \sum_{k=1}^{N} d\lambda_{k}\,(\partial f_{k}/\partial R^{\alpha\beta}), \qquad d\mu^{*\gamma} = \sum_{k=1}^{N} d\lambda_{k}\,(\partial f_{k}/\partial R_{\gamma}),$$

$$R^{\alpha\beta} = -\,\partial u/\partial\varepsilon^{*}_{\alpha\beta}, \quad R_{\gamma} = -\,\partial u/\partial\mu^{*\gamma},$$

where $d\lambda_{k} \geqslant 0$ at $f_{k} = 0$ and $d\lambda_{k} = 0$ at $f_{k} \neq 0$.

We must also fix the loading functions $f_{k} = f_{k}\,(R^{\alpha\beta},\ R_{\gamma},\ \varepsilon^{*}_{\alpha\beta},\ \mu^{*\gamma},\ T,\ \chi_{s})$ and give equations for determining the χ_{s}. One way to do this is to write

$$\begin{aligned} f_{1} &= K_{\alpha\beta\gamma\delta} R^{\alpha\beta} R^{\gamma\delta} - K_{1}\,(\chi_{1},\ T), \quad d\chi_{1} = R^{\alpha\beta}\, d\varepsilon^{*}_{\alpha\beta}, \\ f_{2} &= K^{\alpha\beta}\,(R_{\alpha} + 2\Gamma_{\delta\nu\gamma\alpha}\mu^{*\gamma} R^{\nu\delta})\,(R_{\beta} + 2\Gamma_{\delta\nu\gamma\beta}\mu^{*\gamma} R^{\nu\delta}) \\ &\quad - K_{2}\,(\chi_{2},\ T). \end{aligned} \tag{4.5.3}$$

Here χ_{2} is a parameter related to the initial magnetization curve, which is given by the equations $R^{\alpha\beta} = 0$ and $R_{\gamma} = q\mu^{*}_{\gamma}$, and the increments $d\chi_{2}$ are determined via (4.4.9) with $k = 2$. The coefficients $K_{\alpha\beta\gamma\delta}$, $K^{\alpha\beta}$, $\Gamma_{\alpha\beta\gamma\nu}$, and q belong to the set $K_{(B)}$ and the functions $K_{1}\,(\chi_{1},\ T)$ and $K_{2}\,(\chi_{2},\ T)$ are considered known.

Let us define the body and surface density of the "magnetic charge" thus:

$$\rho z_{(\mathrm{m})} = (1/4\pi)\, \nabla_{\alpha} H^{\alpha}, \qquad \sigma_{(\mathrm{m})} = (1/4\pi)\, n_i\, (H_1^i - H_2^i),$$

where $\mathbf{n}$ is the unit vector normal to the discontinuity surface of the magnetic field strength and directed from side 2 of this surface to side 1. Then the total magnetic charge of any body is identically zero, and the momentum equation can be written as

$$\begin{aligned}
&\rho D v^k = \nabla_r\, [\rho\, (\partial u/\partial \varepsilon_{\alpha\beta})\, x_{\alpha}^k x_{\beta}^r + \tau^{kr}] + f_{(\mathrm{m})}^k,\\
&f_{(\mathrm{m})}^k = \nabla_r\, [(1/4\pi)\, H^k H^r - (1/8\pi)\, H^i H_i g^{kr}] \qquad\qquad (4.5.4)\\
&= \rho z_{(\mathrm{m})} H^k + (1/c)\, \varepsilon^{kpq} j_p H_q.
\end{aligned}$$

For $z_{(\mathrm{m})}$ we have

$$\begin{aligned}
z_{(\mathrm{m})} &= -\frac{1}{\rho}\, \nabla_{\alpha} \rho \mu^{\alpha} = -\frac{1}{\rho \sqrt{|g_{\alpha\beta}|}}\, \frac{\partial}{\partial \xi^{\alpha}}\, (\rho \sqrt{|g_{\alpha\beta}|}\, \mu^{\alpha})\\
&= -\frac{1}{\mathring{\rho} \sqrt{|\mathring{g}_{\alpha\beta}|}}\, \frac{\partial}{\partial \xi^{\gamma}}\, (\mathring{\rho} \sqrt{|\mathring{g}_{\alpha\beta}|}\, \mu^{\gamma}),
\end{aligned}$$

which implies that the magnetic charge of an individual particle in the medium can change only when it reaches the loading surface in the space of variables $\{R^{\alpha\beta}, R_{\gamma}\}$, if $\mu_{(\mathrm{e})}^{\alpha} \equiv 0$ and $\mu^{\gamma} = \mu^{*\gamma}$.

If we take into account the definition of reversible magnetization, we can assume without loss of generality that $u = u\, (S,\ \varepsilon_{\alpha\beta},\ \varepsilon_{\alpha\beta}^{*},\ \breve{\mu}_{\mathrm{e})}^{\gamma},\ \mu^{*\gamma},\ K_{(B)})$. Then the Euler variational equations for μ^{γ} and $\mu^{*\lambda}$ become

$$H_{\gamma} - \partial u/\partial \mu_{(\mathrm{e})}^{\gamma} = h_{\gamma}, \quad H_{\alpha} - \partial u/\partial \mu^{*\alpha} - h_{\alpha} = R_{\alpha}.$$

For a medium in which reversible magnetization is zero, i.e. $\mu_{(\mathrm{e})}^{\alpha} = 0$, the main equations simplify since in this case $\mu^{*\alpha} = \mu^{\alpha}$. If, in addition, the hysteresis loop is a rectangle, then $\partial u/\partial \mu^{*\alpha} = 0$ and, hence, $u = u\, (S,\ \varepsilon_{\alpha\beta},\ \varepsilon_{\alpha\beta}^{*},\ K_{(B)})$. Then the components of the tensor $\rho\, (\partial u/\partial \varepsilon_{\alpha\beta})$ coincide with the components of the internal stress tensor for the usual elastoplastic medium.

4.5.3. The Model of a Magnetoelastic Body with Plastic Deformations, Magnetic Hysteresis, Gyromagnetic Effects, and the Magnetic Inhomogeneity Energy. In this case the Lagrangian and δW^{*} are given by (4.2.15), (4.2.14), and (4.2.4).

The complete set of equations consists of Eqs. (4.2.17)-(4.2.22), the kinetic equation (4.3.9), and the law (4.3.6), while instead of the energy equation (4.2.21) we may take the entropy balance equation. Besides, we must fix the dependence of the loading functions of their arguments and the law of variation of the parameters χ_s. This can be done via Eqs. (4.5.3).

The relations obtained above simplify and become linear if we assume that gradients of the displacement and magnetization vectors are small.

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5

Magnetizable and Polarizable Media with Microstructure

V. A. Zhelnorovich, *Cand. Sc. (Physics and Mathematics)*
Institute of Mechanics, M. V. Lomonosov Moscow State University

INTRODUCTION

In this article we consider models of magnetizable and polarizable media with a microstructure determined by the Cosserat continuum within the framework of Newtonian mechanics. The Newtonian theory of models of magnetizable and polarizable media based on an energy equation and irreversible thermodynamics was developed by Sedov [1]. Models of magnetizable and polarizable media with a microstructure related to the Cosserat continuum have been studied, for instance, in [2-4].

5.1 THE DETERMINING PARAMETERS OF MAGNETIZABLE AND POLARIZABLE MEDIA WITH MICROSTRUCTURE

We define the law of motion of a continuous medium in terms of functions

$$x^\alpha = x^\alpha (\xi^\beta, t), \tag{5.1.1}$$

where x^α are the variables of the observer's Cartesian system of coordinates with a basis Э_α in three-dimensional Euclidean space[1], ξ^β the variables of a Lagrangian system of coordinates with a basis $\hat{\text{Э}}_\alpha$ (comoving system connected with the medium), and t the absolute time. From the physics of the problem it follows that the functions (5.1.1) are one-to-one functions and can be solved for ξ^β:

$$\xi^\beta = \xi^\beta (x^\alpha, t). \tag{5.1.2}$$

By means of a comoving system of coordinates we can introduce the velocity vector $\mathbf{v}$ of individual points in the medium. In the observer's system of coordinate, this vector has components

$$v^\alpha = dx^\alpha/dt, \tag{5.1.3}$$

with d/dt the symbol for a material time derivative with variables ξ^α kept constant. We will also need the vorticity vector $\omega^\alpha \text{Э}_\alpha$ and the rate-of-strain tensor $e_{\alpha\beta}\text{Э}^{\alpha}\text{Э}^{\beta}$ with components defined by the following relationships:

---

[1] It is understood that Greek indices can take on any value 1, 2, 3.

$$e_{\alpha\beta} = (1/2)\,(\partial_\alpha v_\beta + \partial_\beta v_\alpha), \quad \omega^\alpha = (1/2)\,\mathrm{curl}^\alpha\,\mathbf{v}, \tag{5.1.4}$$

where $\partial_\alpha = \partial/\partial x^\alpha$.

We assume that $g_{\alpha\beta}$ are the covariant components of the metric tensor for the three-dimensional space in the observer's coordinate system and that $\hat{g}_{\mu\nu}$ the covariant components of the metric tensor in the comoving coordinate system. The components $g_{\alpha\beta}$ and $\hat{g}_{\mu\nu}$ are linked by the tensor transformation law:

$$\hat{g}_{\mu\nu} = x^\alpha{}_\mu x^\beta{}_\nu g_{\alpha\beta}, \tag{5.1.5}$$

with $x^\alpha{}_\mu = \partial x^\alpha/\partial\xi^\mu$ the distortion. The strain tensor of the medium is determined in the comoving coordinate system by the components

$$\hat{\varepsilon}_{\mu\nu} = (1/2)\,(\hat{g}_{\mu\nu} - \mathring{g}_{\mu\nu}). \tag{5.1.6}$$

Here $\mathring{g}_{\mu\nu} = \mathring{g}_{\mu\nu}\,(\xi^1, \xi^2, \xi^3)$ are the components of the metric tensor for the space of initial states defined on the manifold $\{\xi^\alpha\}$. We can also define the mass density of the medium, ρ, in terms of the metric tensor components $\hat{g}_{\mu\nu}$:

$$\rho = \rho_0 \hat{g}^{-1/2}, \tag{5.1.7}$$

where $\rho_0 = \rho_0\,(\xi^\alpha)$ is a given function, and $\hat{g} = \det \|\hat{g}_{\mu\nu}\|$.

The definition (5.1.7) shows that the mass density of the medium satisfies the continuity equation:

$$d\rho/dt + \rho\,\mathrm{div}\,\mathbf{v} = 0. \tag{5.1.8}$$

We will describe the electromagnetic field in the medium by the electric field strength **E** and the magnetic induction **B**, which in the observer's inertial frame of reference are given by the components E_α and B_α. These components can be expressed in terms of the vector and scalar potentials thus:

$$B_\alpha = \mathrm{curl}_\alpha \mathbf{A}, \quad E_\alpha = -\partial_\alpha \varphi - (1/c)\,\partial\mathbf{A}/\partial t, \tag{5.1.9}$$

where c is the speed of light in vacuum, and $\partial/\partial t$ the symbol of a partial derivative with respect to time t with the variables x^α kept constant.

Besides the observer's Cartesian inertial reference frame we introduce for each point of the medium at each moment of time an inertial ("proper") rest reference frame. In such a reference frame the velocity of the point is zero and the axes are directed along those of the observer's frame of reference. In the rest frame we define the magnetization and electric polarization of a medium by the vector **m**, which is the specific density of the intrinsic magnetic moment of the medium, and vector $\mathbf{p}$, which is the specific density of the intrinsic electric moment of the medium.

To describe magnetization and polarization of a medium in an arbitrary inertial reference frame connected with the observer, we can introduce various vectors linked to $\mathbf{m}$ and $\mathbf{p}$. The corresponding possible definitions are based on the relativistic theory of electromagnetism. In the relativistic theory of magnetizable and polarizable media, if we wish to stay within the framework of special relativity, we can introduce the four-dimensional tensor of specific density of the medium's magnetization and polarization, which in an inertial Cartesian reference frame is defined by its components m^{ij}. The spatial and temporal components of this tensor form 3-vectors with components $\tilde{m}_\alpha$ and $\tilde{p}^\alpha$, which can be used as the Newtonian characteristics of magnetization and polarization of a medium in the observer's coordinate system.

Note that the 3-vectors with components $\tilde{m}_\alpha$ and $\tilde{p}^\alpha$, which together constitute the four-dimensional tensor with components m^{ij}, depend to a great extent on the observer's inertial reference frame and, therefore, their physical meaning is strongly restricted [5]. However, as we can easily verify, the 3-vectors $\mathbf{m} = m^\alpha \text{Э}_\alpha$ and $\mathbf{p} = p^\alpha \text{Э}_\alpha$, defined by means of the vectors $\tilde{\mathbf{m}} = \tilde{m}^\alpha \text{Э}_\alpha$ and $\tilde{\mathbf{p}} = \tilde{p}^\alpha \text{Э}_\alpha$ in the following manner:

$$\mathbf{m} = \tilde{\mathbf{m}} + (1/c)\,[\mathbf{v}, \tilde{\mathbf{p}}], \quad \mathbf{p} = \tilde{\mathbf{p}} - (1/c)\,[\mathbf{v}, \tilde{\mathbf{m}}], \tag{5.1.10}$$

within the nonrelativistic approximation (in the given case to within terms $(v/c)^2$) are invariant with respect to the choice of the Cartesian inertial reference frame of the observer. In this connection it is more preferable from the standpoint of physics to use the 3-vectors $\mathbf{m}$ and $\mathbf{p}$ as the determining parameters for magnetizable and polarizable media rather than $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{p}}$.

In what follows we will consider models of continuous media with a microstructure determined at each point by an orthonormal basis $\mathbf{e}_a$ $(a = 1, 2, 3)$ related to the basis Э_α in the following manner:

$$\text{Э}_\alpha = h_\alpha{}^a \mathbf{e}_a, \quad \mathbf{e}_a = h^\alpha{}_a \text{Э}_\alpha. \tag{5.1.11}$$

Since both sets of basis vectors are orthonormal, the respective scale factors $h_\alpha{}^a$ and $h^\alpha{}_a$ form orthogonal matrices

$$\| h^\alpha{}_a \|^{-1} = \| h^\alpha{}_a \|^T, \quad \| h_\alpha{}^a \|^{-1} = \| h_\alpha{}^a \|^T, \tag{5.1.12}$$

or

$$\delta_{\alpha\beta} h^\alpha{}_a h^\beta{}_b = \delta_{ab}, \quad \delta^{ab} h^\alpha{}_a h^\beta{}_b = \delta^{\alpha\beta}, \tag{5.1.13}$$

where $\delta_{\alpha\beta}$ and $\delta^{\alpha\beta}$ are Kronecker deltas.

In view of the orthogonality conditions (5.1.13), the scale factors $h^{\alpha}{}_{a}$ contain, in general, only three independent parameters, say the Euler angles defining the basis vectors $\mathbf{e}_a$. As the independent parameters defining transformation (5.1.11), as shown in [2, 6], we may also take the components of the two-component spinor ψ.

In the general case, the basis vectors $\mathbf{e}_a$ may depend on time t, rotating about the fixed basis vectors $\mathbf{Э}_\alpha$. Let us introduce the angular velocity vector $\mathbf{\Omega}$ of the $\mathbf{e}_a$ by

$$\mathbf{\Omega} = \Omega^{\alpha}\mathbf{Э}_{\alpha} = \Omega^{a}\mathbf{e}_{a}, \tag{5.1.14}$$

defined by the vector relationship

$$d\mathbf{e}_a/dt = [\mathbf{\Omega},\ \mathbf{e}_a]. \tag{5.1.15}$$

The components of the angular velocity, Ω^a and Ω_α, are expressed in terms of the scale factor $h^{\alpha}{}_{a}$ thus:

$$\Omega^{a} = -(1/2)\ \varepsilon^{abc}\delta_{\alpha\beta}h^{\alpha}{}_{b}\ dh^{\beta}{}_{c}/dt, \tag{5.1.16}$$
$$\Omega_{\alpha} = (1/2)\ \varepsilon_{\alpha\beta\lambda}\delta^{ab}h^{\beta}{}_{a}\ dh^{\lambda}{}_{b}/dt,$$

where ε^{abc} and $\varepsilon_{\alpha\beta\lambda}$ are totally antisymmetric symbols in three dimensions.

Definitions (5.1.16) yield a formula for the time-derivative of the scale factors:

$$dh^{\alpha}{}_{a}/dt = \varepsilon^{\alpha}{}_{\beta\lambda}\Omega^{\beta}h^{\lambda}{}_{a}. \tag{5.1.17}$$

The change in the orthonormal basis vectors $\mathbf{e}_a$ resulting from moving from one point in space to another is determined by the Ricci rotation coefficients, i.e.

$$d\mathbf{e}_a = dx^{\alpha}\ \Delta_{\alpha,\,ab}\mathbf{e}^{b}, \tag{5.1.18}$$

which are expressed in terms of the scale factors thus:

$$\Delta_{\alpha,\,ab} = (1/2)\ \delta_{\lambda\theta}\ (h^{\lambda}{}_{b}\ \partial_{\alpha}h^{\theta}{}_{a} - h^{\lambda}{}_{a}\ \partial_{\alpha}h^{\theta}{}_{b}). \tag{5.1.19}$$

Along with the Ricci rotation coefficients $\Delta_{\alpha,\,ab}$ we also use the quantity $\Delta_{\alpha}{}^{a}$, which is related to $\Delta_{\alpha,\,ab}$ thus:

$$\Delta_{\alpha}{}^{a} = (1/2)\ \varepsilon^{abc}\ \Delta_{\alpha,\,bc},\quad \Delta_{\alpha,\,ab} = \varepsilon_{abc}\ \Delta_{\alpha}{}^{c}. \tag{5.1.20}$$

This enables us to define the tensor

$$\Delta = \Delta_{\beta}{}^{\alpha}\mathbf{Э}^{\beta}\mathbf{e}_{a} = \Delta_{\beta\alpha}\mathbf{Э}^{\beta}\mathbf{Э}^{\alpha}, \tag{5.1.21}$$

whose components are expressible in terms of the scale factors:

$$\Delta_{\beta\alpha} = (1/2)\ \varepsilon_{\alpha\lambda\eta}\ \delta^{ab}h^{\lambda}{}_{a}\ \partial_{\beta}h^{\eta}{}_{b}. \tag{5.1.22}$$

Since the 12 quantities Ω_α and $\Delta_{\beta\alpha}$ depend only on three independent arbitrary parameters contained in the scale factors $h^{\alpha}{}_{a}$, it is clear that the Ω_α and $\Delta_{\beta\alpha}$ are not independent; we can easily obtain the various relationships. Indeed, it follows directly from definition (5.1.22) that always

$$\partial_\theta \Delta_{\beta\alpha} - \partial_\beta \Delta_{\theta\alpha} = \varepsilon_\alpha{}^{\mu\lambda} \Delta_{\theta\mu} \Delta_{\beta\lambda}, \qquad (5.1.23)$$

which is equivalent to the condition that the curvature tensor of the three-dimensional physical space be equal to zero. The above identity contains no more than six independent identities.

If we take the time derivatives of the $\Delta_{\beta\alpha}$ defined in (5.1.22), allowing for (5.1.17), we can find an identity that connects the angular velocity components Ω_α with the tensor components $\Delta_{\beta\alpha}$:

$$d\Delta_{\beta\alpha}/dt - \partial_\beta \Omega_\alpha = \varepsilon_\alpha{}^{\mu\lambda} \Omega_\mu \Delta_{\beta\lambda} - \Delta_{\lambda\alpha} \partial_\beta v^\lambda. \qquad (5.1.24)$$

The vector $\mathbf{\Omega}$ defining the angular velocity of rotation of the basis vectors $\mathbf{e}_a$ and the tensor Δ defining the "turn" of the $\mathbf{e}_a$ will be called, when used in physical applications, the vector of intrinsic rotation and the tensor of intrinsic turns.

5.2 DYNAMICAL EQUATIONS

Let us write out the set of differential equations describing models of magnetizable and polarizable media with microstructure [2, 4].

(i) *The Maxwell equations*:

$$\begin{gathered}\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{E} = -(1/c)\, \partial \mathbf{B}/\partial t, \\ \operatorname{curl} \mathbf{H} = (1/c)\, \partial \mathbf{D}/\partial t + (4\pi/c)\, \mathbf{i}, \quad \operatorname{div} \mathbf{D} = 4\pi\rho_e.\end{gathered} \qquad (5.2.1)$$

Here $\mathbf{D} = \mathbf{E} + 4\pi\rho\tilde{\mathbf{p}}$ is the electric displacement, $\mathbf{H} = \mathbf{B} - 4\pi\rho\tilde{\mathbf{m}}$ the magnetic field strength, ρ the mass density of the medium, ρ_e the free charge density of the medium, and $\mathbf{i}$ the conduction current vector not associated with the motion of the medium's electric charge ρ_e.

(ii) *The magnetization equations*:

$$B^\alpha - \frac{1}{c} [\mathbf{v}, \mathbf{E}]^\alpha + \frac{1}{\rho} \left(\frac{\partial \Lambda_0}{\partial m_\alpha} - \partial_\lambda \frac{\partial \Lambda_0}{\partial \partial_\lambda m_\alpha} \right) = L^\alpha. \qquad (5.2.2)$$

Here L^α are the components of a vector, to be defined later, that determines magnetization relaxation processes in the medium, and Λ_0 is a given function of the following arguments[2]:

$$x^\alpha{}_\beta,\ \partial_\beta v_\alpha,\ m_\alpha,\ \partial_\beta m_\alpha,\ p_\alpha,\ \partial_\beta p_\alpha,\ \Omega_\alpha,\ \Delta_{\beta\alpha},\ s,\ K_B, \qquad (5.2.3)$$

---

[2] The set of equations (5.2.2) has been obtained in [2, 4] via a variational equation for the case where the velocity gradients $\partial_\beta v_\alpha$ enter Λ_0 through the vorticity components ω_α. The function Λ_0 is a part of the Lagrangian Λ of the given model, i.e.

$$\Lambda = (1/2)\, \rho v^2 + (1/8\pi)\, (E^2 - B^2) + \rho\, (B_\alpha \tilde{m}^\alpha + E_\alpha \tilde{p}^\alpha) + \Lambda_0.$$

where s is the specific entropy density, and the K_B are parameters given as functions of coordinates and determining the various physical and geometric properties of the medium (e.g. its anisotropy).

(iii) *The polarization equations*:

$$E^\alpha + \frac{1}{c}[\mathbf{v}, \mathbf{B}]^\alpha + \frac{1}{\rho}\left(\frac{\partial\Lambda_0}{\partial p_\alpha} - \partial_\lambda \frac{\partial\Lambda_0}{\partial\partial_\lambda p_\alpha}\right) = \Pi^\alpha. \tag{5.2.4}$$

Here Π^α are the components of a vector, to be defined later, that determines polarization relaxation processes in the medium.

(iv) *The moment of momentum equations*:

$$\rho\, dk^\alpha/dt + \partial_\beta D^{\alpha\beta} + \varepsilon^\alpha{}_{\lambda\mu}\left[(\partial\Lambda_0/\partial\Omega_\lambda)\,\Omega^\mu + (\partial\Lambda_0/\partial\Delta_{\beta\lambda})\,\Delta_\beta{}^\mu\right]$$
$$= \overset{*}{R}{}^\alpha. \tag{5.2.5}$$

Here $\overset{*}{R}{}^\alpha$ is the relaxation term, k^α the components of the tensor of the specific density of the intrinsic moment of momentum of the medium, and $D^{\alpha\beta}$ the components of the couple stress tensor:

$$k^\alpha = \frac{\partial\Lambda_0/\rho}{\partial\Omega_\alpha}, \quad D^{\alpha\beta} = \frac{\partial\Lambda_0}{\partial\Delta_{\beta\alpha}} + A^{\alpha\beta}. \tag{5.2.6}$$

The relaxation term $\overset{*}{R}{}^\alpha$ and the part of the couple stress tensor defined by the components $A^{\alpha\beta}$ are given on the basis of thermodynamic considerations.

(v) *The momentum equation*:

$$\rho\,\frac{d\,(g_{\alpha(m)}/\rho)}{dt} = \partial_\beta P_{\alpha(m)}^{\beta} + Q_\alpha$$
$$+ \frac{1}{8\pi}\left(B_\lambda\partial_\alpha H^\lambda - H^\lambda\partial_\alpha B_\lambda + D_\lambda\partial_\alpha E^\lambda - E^\lambda\partial_\alpha D_\lambda\right)$$
$$+ \frac{\rho}{c}\,\varepsilon_{\alpha\beta\lambda}\,\frac{d}{dt}\left(E^\beta m^\lambda - B^\beta p^\lambda\right) + \rho_e E_\alpha$$
$$+ \frac{1}{c}\,\varepsilon_{\alpha\beta\lambda}\left(i^\beta + \rho_e v^\beta\right) B^\lambda. \tag{5.2.7}$$

Here Q_α are the components of the external body force acting on the medium, $g_{\alpha(m)}$ the components of the body density of the momentum of the medium, and $P_{\alpha(m)}^{\beta}$ the components of the stress tensor of the medium:

$$
\begin{aligned}
P_{\alpha(m)}^{\beta} = &-\frac{\partial \Lambda_0}{\partial x^{\alpha}{}_{\lambda}} x^{\beta}{}_{\lambda} + \frac{\partial \Lambda_0}{\partial \partial_{\beta} p_{\lambda}} \partial_{\alpha} p_{\lambda} \\
&+ \frac{\partial \Lambda_0}{\partial \partial_{\beta} m_{\lambda}} \partial_{\alpha} m_{\lambda} + \frac{\partial \Lambda_0}{\partial \partial_{\beta} v_{\lambda}} \partial_{\alpha} v_{\lambda} + \frac{\partial \Lambda_0}{\partial \Delta_{\beta\lambda}} \Delta_{\alpha\lambda} \\
&+ \delta_{\alpha}^{\beta} \left\{ -\Lambda_0 + \frac{1}{2} \rho m_{\lambda} \left(B^{\lambda} - \frac{1}{c} [\mathbf{v}, \mathbf{E}]^{\lambda} \right) \right. \\
&\left. + \frac{1}{2} \rho p_{\lambda} \left(E^{\lambda} + \frac{1}{c} [\mathbf{v}, \mathbf{B}]^{\lambda} \right) \right\} - \frac{1}{2} \varepsilon_{\alpha}{}^{\beta\lambda} \overset{*}{R}_{\lambda} \qquad (5.2.8) \\
&+ \frac{1}{2} \rho \left(m_{\alpha} L^{\beta} - m^{\beta} L_{\alpha} + p_{\alpha} \Pi^{\beta} - p^{\beta} \Pi_{\alpha} \right) + \tau_{\alpha}{}^{\beta},
\end{aligned}
$$

$$
g_{\alpha(m)} = \rho v_{\alpha} - \partial_{\lambda} \frac{\partial \Lambda_0}{\partial \partial_{\lambda} v^{\alpha}}.
$$

The tensor components $\tau_{\alpha}{}^{\beta}$ in (5.2.8) determine the viscous stress tensor of the medium. The contravariant components of the viscous stress tensor are symmetric, by definition: $\tau^{\alpha\beta} = \tau^{\beta\alpha}$.

(vi) *The continuity equations for the mass density and electric charge density of the medium:*

$$
\partial \rho / \partial t + \operatorname{div} \rho \mathbf{v} = 0, \quad \partial \rho_e / \partial t + \operatorname{div} (\rho_e \mathbf{v} + \mathbf{i}) = 0. \qquad (5.2.9)
$$

(vii) *The temperature equation:*

$$
\rho T + \partial \Lambda_0 / \partial s = 0. \qquad (5.2.10)
$$

(viii) *The entropy balance equation:*

$$
\begin{aligned}
\rho T\, ds/dt = &-\partial_{\alpha} q^{\alpha} + \tau^{\alpha\beta} e_{\alpha\beta} - A^{\alpha\beta} \partial_{\beta} \Omega_{\alpha} \\
&+ i^{\beta} (E_{\beta} + (1/c) [\mathbf{v}, \mathbf{B}]_{\beta}) - \overset{*}{R}{}^{\alpha} (\Omega_{\alpha} - \omega_{\alpha}) \\
&+ \rho L^{\alpha} (dm_{\alpha}/dt - [\boldsymbol{\omega}, \mathbf{m}]_{\alpha}) + \rho \Pi^{\alpha} (dp_{\alpha}/dt - [\boldsymbol{\omega}, \mathbf{p}]_{\alpha}). \qquad (5.2.11)
\end{aligned}
$$

Here q^{α} are the components of the heat flux vector.

We define the inner entropy production $d_i s/dt$ by the equation

$$
\rho\, ds/dt = -T \partial_{\beta} (q^{\beta}/T) + \rho\, d_i s/dt, \qquad (5.2.12)
$$

comparing Eqs. (5.2.12) and (5.2.11), we obtain

$$
\begin{aligned}
\rho T\, d_i s/dt = &-(q^{\alpha}/T) \partial_{\alpha} T + \tau^{\alpha\beta} e_{\alpha\beta} - A^{\alpha\beta} \partial_{\beta} \Omega_{\alpha} \\
&+ i^{\beta} (E_{\beta} + (1/c) [\mathbf{v}, \mathbf{B}]_{\beta}) - \overset{*}{R}{}^{\alpha} (\Omega_{\alpha} - \omega_{\alpha}) \\
&+ \rho L^{\alpha} (dm_{\alpha}/dt - [\boldsymbol{\omega}, \mathbf{m}]_{\alpha}) + \rho \Pi^{\alpha} (dp_{\alpha}/dt - [\boldsymbol{\omega}, \mathbf{p}]_{\alpha}). \qquad (5.2.13)
\end{aligned}
$$

The quantities

$$
q^{\alpha}, \ \tau^{\alpha\beta}, \ A^{\alpha\beta}, \ i^{\beta}, \ \overset{*}{R}{}^{\alpha}, \ L^{\alpha}, \ \Pi^{\alpha}, \qquad (5.2.14)
$$

appearing in the expression for the inner entropy production, are usually called thermodynamic fluxes, while the quantities

$$\partial_\alpha T,\ e_{\alpha\beta},\ \partial_\beta\Omega_\alpha,\ E_\beta+\frac{1}{c}[\mathbf{v},\ \mathbf{B}]_\beta,\ \Omega_\alpha-\omega_\alpha,$$
$$dm_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{m}]_\alpha,\quad dp_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{p}]_\alpha \tag{5.2.15}$$

are called thermodynamic forces.

In accordance with the second law of thermodynamics,

$$d_is/dt\geqslant 0, \tag{5.2.16}$$

a condition that restricts the possible definitions of the thermodynamic fluxes listed in (5.2.14). To define these quantities and to satisfy (5.2.16), one can employ the dissipation function.

Obviously, if the thermodynamic fluxes are given in the form of functions of thermodynamic forces and, possibly, other determining parameters of medium and field, so is the quantity $\sigma=\rho\, d_is/dt$:

$$\sigma=\rho\, d_is/dt=\sigma\,(e_{\alpha\beta},\ \partial_\alpha T,\ \partial_\beta\Omega_\alpha,\ E_\beta+(1/c)\,[\mathbf{v},\ \mathbf{B}]_\beta,$$
$$\Omega_\alpha-\omega_\alpha,\ dm_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{m}]_\alpha,\ dp_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{p}]_\alpha). \tag{5.2.17}$$

In this case σ is called the dissipation function.

In many cases we can assume that thermodynamic fluxes are expressible in terms of the derivatives of the dissipation function with respect to thermodynamic forces, namely

$$q^\alpha=\lambda_1\,\partial\sigma/\partial\partial_\alpha T,\quad \tau^{\alpha\beta}=\lambda_2\,\partial\sigma/\partial e_{\alpha\beta},$$
$$i^\beta=\lambda_3\,\partial\sigma/\partial\,(E_\beta+(1/c)\,[\mathbf{v},\ \mathbf{B}]_\beta),$$
$$A^{\alpha\beta}=\lambda_4\,\partial\sigma/\partial\partial_\beta\Omega_\alpha, \tag{5.2.18}$$
$$\overset{*}{R}{}^\alpha=\lambda_5\,\partial\sigma/\partial\,(\Omega_\alpha-\omega_\alpha),\quad L^\alpha=\lambda_6\,\partial\sigma/\partial\,(dm_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{m}]_\alpha),$$
$$\Pi^\alpha=\lambda_7\,\partial\sigma/\partial\,(dp_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{p}]_\alpha),$$

with $\lambda_1,\ \ldots,\ \lambda_7$ scalar functions of the determining parameters of medium and field.

If the thermodynamic forces $\partial_\alpha T$, $e_{\alpha\beta}$, etc. in (5.2.13) are expressible in terms of the thermodynamic fluxes q^α, $\tau^{\alpha\beta}$, etc., the dissipation function can be written in the form of a function of thermodynamic fluxes:

$$\sigma=\sigma\,(q^\alpha,\ \tau^{\alpha\beta},\ i^\beta,\ A^{\alpha\beta},\ \overset{*}{R}{}^\alpha,\ L^\alpha,\ \Pi^\alpha,\ \ldots), \tag{5.2.19}$$

and instead of (5.2.18) we can write the following:

$$\partial_\alpha T=\mu_1\,\partial\sigma/\partial q^\alpha,\quad e_{\alpha\beta}=\mu_2\,\partial\sigma/\partial\tau^{\alpha\beta},$$
$$E_\alpha+(1/c)\,[\mathbf{v},\ \mathbf{B}]_\alpha=\mu_3\,\partial\sigma/\partial i^\alpha,\quad \partial_\beta\Omega_\alpha=\mu_4\,\partial\sigma/\partial A^{\alpha\beta},$$
$$\Omega_\alpha-\omega_\alpha=\mu_5\,\partial\sigma/\partial\overset{*}{R}{}^\alpha,\quad dm_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{m}]_\alpha=\mu_6\,\partial\sigma/\partial L^\alpha, \tag{5.2.20}$$
$$dp_\alpha/dt-[\boldsymbol{\omega},\ \mathbf{p}]_\alpha=\mu_7\,\partial\sigma/\partial\Pi^\alpha,$$

with $\mu_1,\ \ldots,\ \mu_7$ scalar functions of the determining parameters.

We could also take the case where σ is given as a function of part of the thermodynamic forces and part of the thermodynamic fluxes. Then to complete the set of equations we would have to take the respective parts of (5.2.18) and (5.2.20).

If the dissipation function is quadratic, the thermodynamic fluxes and forces are linked via linear equations, which are commonly known as the Onsager relations. As applied to the theory under discussion we can write the Onsager relations in the simplest case as

$$\begin{gathered}
q^{\alpha} = -\varkappa^{\alpha\beta}\,\partial_{\beta}T, \quad \tau^{\alpha\beta} = \tau^{\alpha\beta\lambda\theta}e_{\lambda\theta}, \\
l^{\alpha}_{\beta}L^{\beta} = \eta^{\alpha\beta}\rho\,(dm_{\beta}/dt - [\boldsymbol{\omega},\ \mathbf{m}]_{\beta}), \\
\overset{*}{R}{}^{\alpha} = -r^{\alpha\beta}\,(\Omega_{\beta} - \omega_{\beta}), \qquad (5.2.21) \\
\pi^{\alpha}_{\beta}\Pi^{\beta} = n^{\alpha\beta}\rho\,(dp_{\beta}/dt - [\boldsymbol{\omega},\ \mathbf{p}]_{\beta}), \quad A^{\alpha\beta} = -a^{\alpha\beta\lambda\theta}\,\partial_{\lambda}\Omega_{\theta}, \\
i^{\beta} = \sigma^{\beta\lambda}\,(E_{\lambda} + (1/c)\,[\mathbf{v},\ \mathbf{B}]_{\lambda}).
\end{gathered}$$

Here the factors $\varkappa$, τ, l, η, a, π, σ, n, and r may be functions of the determining parameters of medium and field and are chosen in such a way as to satisfy condition (5.2.16). If the components π^{α}_{β} (or δ^{α}_{β}) in Eqs. (5.2.21) constitute a nonsingular matrix, then without restricting generality we can assume that $\pi^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ (or, respectively, $l^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$).

Note that we can write the expression for the inner entropy production, (5.2.13), in an equivalent form:

$$\begin{aligned}
\rho T\,d_{i}s/dt &= -(q^{\alpha}/T)\,\partial_{\alpha}T + \tau^{\alpha\beta}e_{\alpha\beta} - A^{\alpha\beta}\,\partial_{\beta}\Omega_{\alpha} \\
&+ i^{\beta}\,(E_{\beta} + (1/c)\,[\mathbf{v},\ \mathbf{B}]_{\beta}) - [\overset{*}{R}{}^{\alpha} - \varepsilon^{\alpha\mu\nu}\rho\,(m_{\mu}L_{\nu} + p_{\mu}\Pi_{\nu})] \\
&\times (\Omega_{\alpha} - \omega_{\alpha}) + \rho L^{\alpha}\,(dm_{\alpha}/dt - [\boldsymbol{\Omega},\ \mathbf{m}]_{\alpha}) \\
&+ \rho\Pi^{\alpha}\,(dp_{\alpha}/dt - [\boldsymbol{\Omega},\ \mathbf{p}]_{\alpha}). \qquad (5.2.22)
\end{aligned}$$

If we take $d\mathbf{m}/dt - [\boldsymbol{\Omega},\ \mathbf{m}]$ and $d\mathbf{p}/dt - [\boldsymbol{\Omega},\ \mathbf{p}]$ as thermodynamic forces, then instead of (5.2.21) we can write the following Onsager relations:

$$\begin{gathered}
l^{\alpha}_{\beta}L^{\beta} = \eta^{\alpha\beta}\rho\,(dm_{\beta}/dt - [\boldsymbol{\Omega},\ \mathbf{m}]_{\beta}), \quad \pi^{\alpha}_{\beta}\Pi^{\beta} = n^{\alpha\beta}\rho\,(dp_{\beta}/dt - [\boldsymbol{\Omega},\ \mathbf{p}]_{\beta}), \\
\overset{*}{R}{}^{\alpha} - [\rho\mathbf{m},\ \mathbf{L}]^{\alpha} - [\rho\mathbf{p},\ \boldsymbol{\Pi}]^{\alpha} = -r^{\alpha\beta}\,(\Omega_{\beta} - \omega_{\beta}). \qquad (5.2.23)
\end{gathered}$$

These are used in some theories of magnetizable liquids.

The differential equations (5.2.1)-(5.2.11) and (5.2.18) together with the kinematic equations (5.1.23) and (5.1.24) generally form a closed set of equations describing models of magnetizable and polarizable media in an electromagnetic field. The physical concretization of a model is achieved by fixing the function Λ_0 and the relationships defining the thermodynamic fluxes q^{α}, $\tau^{\alpha\beta}$, $A^{\alpha\beta}$, i^{β}, $\overset{*}{R}{}^{\alpha}$, L^{α}, and Π^{α}. For different functions Λ_0 and different relationships defining the thermodynamic fluxes, Eqs. (5.2.1)-(5.2.11) and (5.2.18) describe various models of elastic and liquid, perfect and viscous, magnetizable and polarizable media interacting with an electromagnetic field. Some of these models are discussed below (see also [1-14]).

5.3 RELAXATION MODELS OF MAGNETIZABLE AND POLARIZABLE MEDIA WITHOUT MICROSTRUCTURE

Take the case where Λ_0 has the form[3]

$$\Lambda_0 = -\rho U_m ({x^{\alpha}}_{\beta},\ m_{\alpha},\ p_{\alpha},\ s,\ K_B) - \gamma \rho m^{\alpha} \omega_{\alpha}, \tag{5.3.1}$$

where γ is a constant. Assuming that $A^{\alpha\beta}$ and $\overset{*}{R}{}^{\alpha}$, quantities characterizing dissipation processes associated with the medium's microstructure, are zero and that the relaxation terms L^{α} and Π^{α} are given by the Onsager relations (5.2.21) at $l^{\alpha}_{\beta} = \pi^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$, we arrive at a set of equations that determines a class of models of magnetizable and polarizable media and allows for relaxation of magnetization and polarization:

$$\begin{gathered} B^{\alpha} - (1/c)\,[\mathbf{v}, \mathbf{E}]^{\alpha} = \partial U_m/\partial m_{\alpha} + \gamma \omega^{\alpha} + \eta^{\alpha\beta}\rho\,(dm_{\beta}/dt - [\boldsymbol{\omega}, \mathbf{m}]_{\beta}), \\ E^{\alpha} + (1/c)\,[\mathbf{v}, \mathbf{B}]^{\alpha} = \partial U_m/\partial p_{\alpha} + n^{\alpha\beta}\rho\,(dp_{\beta}/dt - [\boldsymbol{\omega}, \mathbf{p}]_{\beta}), \\ \rho T\, ds/dt = -\partial_{\alpha} q^{\alpha} + \rho^2 \eta^{\alpha\beta}\,(dm_{\alpha}/dt - [\boldsymbol{\omega}, \mathbf{m}]_{\alpha}) \\ \times (dm_{\beta}/dt - [\boldsymbol{\omega}, \mathbf{m}]_{\beta}) + \rho^2 n^{\alpha\beta}\,(dp_{\alpha}/dt \\ - [\boldsymbol{\omega}, \mathbf{p}]_{\alpha})\,(dp_{\beta}/dt - [\boldsymbol{\omega}, \mathbf{p}]_{\beta}) + i^{\beta}\,(E_{\beta} + (1/c)\,[\mathbf{v}, \mathbf{B}]_{\beta}), \\ T = \partial U_m/\partial s, \quad \partial\rho/\partial t + \operatorname{div} \rho\mathbf{v} = 0, \quad \partial\rho_e/\partial t + \operatorname{div}(\rho_e \mathbf{v} + \mathbf{i}) = 0. \end{gathered} \tag{5.3.2}$$

To make the set of equations (5.3.2) complete, we must add to it the Maxwell equations (5.2.1) and the momentum equations (5.2.7), in which the components of the momentum density tensor and of the stress tensor of the medium are

$$\begin{gathered} g_{\alpha(m)} = \rho v_{\alpha} - (1/2)\,\gamma\, \mathrm{curl}_{\alpha} \rho \mathbf{m}, \\ P_{\alpha\binom{\beta}{m}} = \rho\,(\partial U_m/\partial {x^{\alpha}}_{\lambda})\, {x^{\beta}}_{\lambda} + (1/2)\,\gamma \varepsilon^{\beta\lambda\mu} \rho m_{\lambda} \partial_{\alpha} v_{\mu} \\ + (1/2)\,\delta^{\beta}_{\alpha}\,\{\rho m_{\lambda}\,(B^{\lambda} - (1/c)\,[\mathbf{v}, \mathbf{E}]^{\lambda}) \\ + \rho p_{\lambda}\,(E^{\lambda} + (1/c)\,[\mathbf{v}, \mathbf{B}]^{\lambda})\} \\ + (1/2)\,\rho\,(m_{\alpha} L^{\beta} - m^{\beta} L_{\alpha} + p_{\alpha} \Pi^{\beta} - p^{\beta} \Pi_{\alpha}) + {\tau_{\alpha}}^{\beta}. \end{gathered} \tag{5.3.3}$$

From the momentum density in (5.3.3) we see that if we allow for the gyromagnetic energy, the medium's momentum density is determined by the sum of the common term $\rho\mathbf{v}$, associated with the medium's macroscopic motion, and a term containing the curl of magnetization, $-(1/2)\,\gamma$ curl $\rho\mathbf{m}$. The presence of an intrinsic momentum in the medium, $-(1/2)\,\gamma$ curl $\rho\mathbf{m}$, is a characteristic feature of the class of models being considered and has no analog in classical models.

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[3] The last term in Λ_0 represents the gyromagnetic energy of the medium.

If the $x^\alpha{}_\beta$ and K_B enter U_m only through the mass density ρ, then Eqs. (5.3.2) and (5.3.3) define models of magnetizable and polarizable liquids. In this case the stress tensor components of the medium are

$$P_{\alpha\binom{\beta}{m}} = -p\delta^\beta_\alpha + \tau_\alpha{}^\beta + (1/2)\,\gamma\varepsilon^{\beta\lambda\mu}\rho m_\lambda\,\partial_\alpha v_\mu$$
$$+ (1/2)\,\rho\,(m_\alpha L^\beta - m^\beta L_\alpha + p_\alpha \Pi^\beta - p^\beta \Pi_\alpha), \tag{5.3.4}$$

where

$$p = \rho^2\,\partial U_m/\partial\rho - (1/2)\,\rho\,\{m_\alpha\,(B^\alpha - (1/c)\,[\mathbf{v},\ \mathbf{E}]^\alpha)$$
$$+ p_\alpha\,(E^\alpha + (1/c)\,[\mathbf{v},\ \mathbf{B}]^\alpha)\}. \tag{5.3.5}$$

If we define U_m in (5.3.1) by a formula of the type

$$\rho U_m = \rho U^0_m + \frac{2\pi\mu\,(\rho,\ s)}{\mu\,(\rho,\ s) - 1}\,M_\alpha M^\alpha + \frac{2\pi}{\varepsilon\,(\rho,\ s) - 1}\,P_\alpha P^\alpha, \tag{5.3.6}$$

where $U^0_m\,(\rho,\ s)$, $\mu\,(\rho,\ s)$, and $\varepsilon\,(\rho,\ s)$ are given functions, $M_\alpha = \rho m_\alpha$ the components of the body magnetization density vector of the medium, and $P_\alpha = \rho p_\alpha$ the components of the body polarization density vector of the medium, then the equations for the medium magnetization and polarization in (5.3.2) become

$$E^\alpha + \frac{1}{c}\,[\mathbf{v},\ \mathbf{B}]^\alpha = \frac{4\pi}{\varepsilon - 1}\,P^\alpha + n^{\alpha\beta}\,(\rho d\,(P_\beta/\rho)/dt - [\boldsymbol{\omega},\ \mathbf{P}]_\beta),$$
$$B^\alpha - \frac{1}{c}\,[\mathbf{v},\ \mathbf{E}]^\alpha = \frac{4\pi\mu}{\mu - 1}\,M^\alpha + \gamma\omega^\alpha + \eta^{\alpha\beta}\,(\rho d\,(M_\beta/\rho)/dt \tag{5.3.7}$$
$$- [\boldsymbol{\omega},\ \mathbf{M}]_\beta).$$

In this case, the pressure p in (5.3.4) is

$$p = \rho^2\,\frac{\partial U^0_m}{\partial\rho} - 2\pi\rho\left[\frac{1}{(\mu - 1)^2}\,\frac{\partial\mu}{\partial\rho}\,M_\alpha M^\alpha\right.$$
$$\left. + \frac{1}{(\varepsilon - 1)^2}\,\frac{\partial\varepsilon}{\partial\rho}\,P_\alpha P^\alpha\right] - (1/2)\,(P_\lambda\Pi^\lambda + M_\lambda L^\lambda + \gamma M^\lambda\omega_\lambda). \tag{5.3.8}$$

If the $x^\alpha{}_\beta$ and K_B enter U_m via the strain tensor components $\hat{\varepsilon}_{\alpha\beta}$, Eqs. (5.3.2) and (5.3.3) define models of elastic magnetizable and polarizable media. Here is an example of a closed set of equations describing elastic polarizable media in the electrostatic approximation; we assume in this model that $\Lambda_0 = -\rho U_m\,(\hat{\varepsilon}_{\alpha\beta},\ \hat{P}_\alpha,\ s,\ K_B)$, where $\hat{P}_\alpha = x^\beta{}_\alpha P_\beta$ are the components of the body polarization density vector calculated in the comoving reference frame. The set of equations is

$$\rho\, dv_{\alpha}/dt = \partial_{\beta} P_{\alpha(m)}^{\beta} + (1/2)\,(P_{\lambda}\, \partial_{\alpha} E^{\lambda} - E^{\lambda}\, \partial_{\alpha} P_{\lambda}) + Q_{\alpha},$$
$$E^{\alpha} = \rho x^{\alpha}{}_{\beta}\, \partial U_m/\partial \hat{P}_{\beta} + \Pi^{\alpha},$$
$$\operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div}\,(\mathbf{E} + 4\pi \mathbf{P}) = 0,$$
$$\rho T \frac{ds}{dt} = -\partial_{\alpha} q^{\alpha} + \tau^{\alpha\beta} e_{\alpha\beta} + \Pi^{\alpha} \left(\rho \frac{d}{dt} \frac{1}{\rho} P_{\alpha} - [\boldsymbol{\omega},\ \mathbf{P}]_{\alpha}\right) + i^{\beta} E_{\beta}, \tag{5.3.9}$$
$$T = \partial U_m/\partial s, \quad d\rho/dt + \rho \operatorname{div} \mathbf{v} = 0,$$
$$P_{\alpha(m)}^{\beta} = \rho \xi^{\eta}{}_{\alpha} x^{\beta}{}_{\theta}\, (\hat{P}_{\eta} \partial U_m/\partial \hat{P}_{\theta} + \hat{g}_{\lambda\eta} \partial U_m/\partial \hat{\varepsilon}_{\lambda\theta})$$
$$+ \delta_{\alpha}^{\beta}\, [-(1/2)\, P_{\lambda} E^{\lambda} + P_{\lambda} \Pi^{\lambda}] + \tau_{\alpha}{}^{\beta} + (1/2)\,(P_{\alpha} \Pi^{\beta} - P^{\beta} \Pi_{\alpha}).$$

The relaxation term Π^{α} and the components i^{β} of the conduction current vector are given in the simplest case by the following relationships:

$$\Pi^{\alpha} = n^{\alpha\beta} \left(\rho \frac{d}{dt} \frac{1}{\rho} P_{\beta} - [\boldsymbol{\omega},\ \mathbf{P}]_{\beta}\right), \qquad i^{\beta} = \sigma^{\alpha\beta} E_{\alpha}, \tag{5.3.10}$$

where $n^{\alpha\beta}$ and $\sigma^{\alpha\beta}$ are fixed positive definite quadratic forms. The components $\tau^{\alpha\beta}$ of the viscous stress tensor and the components q^{α} of the heat flux vector may be determined via (5.2.21).

The set of equations (5.3.9) is, in general, extremely complex, but for many practical purposes linearized equations are sufficient.

Let us define the displacement vector $\mathbf{u} = u^{\alpha} \text{Э}_{\alpha}$ of the points in the medium by the quantity $\mathbf{u} = \mathbf{r} - \mathbf{r}_0$, where $\mathbf{r}$ is the radius vector of the points in the medium at a given moment of time, while $\mathbf{r}_0$ is the radius vector of the points of the medium in the initial state. Moreover, we will assume that at the initial moment $t = t_0$ the Cartesian coordinate system of the observer coincides with the comoving system of coordinates. Let us consider the motion of a nonconducting medium in which the displacement vector gradients are small and the deviations of the components P^{α} of the polarization vector, temperature T, entropy s, density of the medium ρ, and the components E^{α} of the electric field vector from the equilibrium (constant) values P_0^{α}, T_0, s_0, ρ_0, and E_0^{α} are small. We put

$$P^{\alpha} = P_0^{\alpha} + \pi^{\alpha}, \quad s = s_0 + s_1, \quad \rho = \rho_0 + \rho_1, \quad E^{\alpha} = E_0^{\alpha} + e^{\alpha},$$

where π^{α}, s_1, ρ_1, and e^{α} are small quantities treated as first-order infinitesimals. We can easily see that to within first-order infinitesimals the following relations are valid:

$$x^{\alpha}{}_{\beta} = \delta_{\beta}^{\alpha} + u^{\alpha}{}_{\beta}, \ \xi^{\alpha}{}_{\beta} = \delta_{\beta}^{\alpha} - u^{\alpha}{}_{\beta}, \ \hat{\varepsilon}_{\alpha\beta} = (1/2)\,(u_{\alpha\beta} + u_{\beta\alpha}),$$
$$\hat{P}_{\alpha} = P_{\alpha}^{0} + P_0^{\beta} u_{\alpha\beta} + \pi_{\alpha},$$

where $u^{\alpha}{}_{\beta} = \partial u^{\alpha}/\partial x^{\beta}$. With this in mind, we can write the linearized equations (5.3.9) in the following form:

$$\operatorname{curl} \mathbf{e} = 0, \quad \partial_{\alpha} (e^{\alpha} + 4\pi\pi^{\alpha}) = 0,$$

$$e_{\alpha} = b^{\alpha\beta}\pi_{\beta} + \zeta^{\mu\nu\alpha}u_{\mu\nu} + f^{\alpha}s_1 + n^{\alpha\beta} (\partial\pi_{\beta}/\partial t - [\boldsymbol{\omega}, \mathbf{P}_0]_{\beta} + P^{0}_{\beta}\, \partial_{\lambda}v^{\lambda}),$$

$$\rho_0 (T - T_0) = \eta^{\alpha\beta}u_{\alpha\beta} + ks_1 + f^{\alpha}\pi_{\alpha}, \quad \rho_0 T_0\, \partial s_1/\partial t = \varkappa^{\alpha\beta}\, \partial_{\alpha}\partial_{\beta}T,$$

$$\rho_0\, \partial^2 u^{\alpha}/\partial t^2 = Q^{\alpha} + \partial_{\beta}\, [\lambda^{\alpha\beta\lambda\theta}u_{\lambda\theta} + \zeta^{\alpha\beta\lambda}\pi_{\lambda} + \eta^{\alpha\beta}s_1 + \tau^{\alpha\beta\lambda\theta}\partial\varepsilon_{\lambda\theta}/\partial t$$

$$+ \delta^{\alpha\beta}P^{0}_{\lambda}\Pi^{\lambda}_{1} + (1/2)\, (P^{\alpha}_{0}\Pi^{\beta}_{1} - P^{\beta}_{0}\Pi^{\alpha}_{1})], \tag{5.3.11}$$

$$\partial\rho_1/\partial t + \rho_0 \operatorname{div} \mathbf{v} = 0, \quad \Pi^{\alpha}_{1} = n^{\alpha\beta} (\partial\pi_{\beta}/\partial t - [\boldsymbol{\omega}, \mathbf{P}_0]_{\beta} + P^{0}_{\beta}\, \partial_{\lambda}v^{\lambda}).$$

The coefficients b, ζ, f, η, k and λ in Eqs. (5.3.11) are expressed in terms of U_m in the following manner:]

$$b^{\alpha\beta} = \left(\frac{\partial^2\rho U_m}{\partial\hat{P}_{\alpha}\, \partial\hat{P}_{\beta}}\right)_0, \quad \zeta^{\mu\nu\alpha} = \left(\frac{\partial^2\rho U_m}{\partial\hat{\varepsilon}_{\mu\nu}\, \partial\hat{P}_{\alpha}} + \hat{P}^{\mu}\frac{\partial^2\rho U_m}{\partial\hat{P}_{\alpha}\, \partial\hat{P}_{\nu}} + \delta^{\alpha\mu}\frac{\partial\rho U_m}{\partial\hat{P}_{\nu}}\right)_0,$$

$$f^{\alpha} = \left(\frac{\partial^2\rho U_m}{\partial\hat{P}_{\alpha}\, \partial s}\right)_0, \quad \eta^{\alpha\beta} = \left(\frac{\partial\rho U_m}{\partial s}\delta^{\alpha\beta} + \frac{\partial^2\rho U_m}{\partial\hat{\varepsilon}_{\alpha\beta}\partial s} + \hat{P}^{\alpha}\frac{\delta^2\rho U_m}{\partial s\, \partial\hat{P}_{\beta}}\right)_0,$$

$$k = \left(\frac{\partial^2\rho U_m}{\partial s^2}\right)_0, \lambda^{\alpha\beta\lambda\theta} = \left(\frac{\partial^2\rho U_m}{\partial\hat{\varepsilon}_{\alpha\beta}\partial\hat{\varepsilon}_{\lambda\theta}} + \hat{P}^{\alpha}\frac{\partial^2\rho U_m}{\partial\hat{\varepsilon}_{\lambda\theta}\;\partial\hat{P}_{\beta}} + \hat{P}^{\lambda}\frac{\partial^2\rho U_m}{\partial\hat{\varepsilon}_{\alpha\beta}\;\partial\hat{P}_{\theta}}\right.$$

$$+ \hat{P}^{\alpha}\hat{P}^{\lambda}\frac{\partial^2\rho U_m}{\partial\hat{P}_{\theta}\, \partial\hat{P}_{\beta}} + \delta^{\alpha\lambda}\frac{\partial\rho U_m}{\partial\hat{\varepsilon}_{\theta\beta}}$$

$$\left. + \delta^{\beta\lambda}\left[\frac{\partial\rho U_m}{\partial\hat{\varepsilon}_{\alpha\theta}} + \hat{P}^{\alpha}\frac{\partial\rho U_m}{\partial\hat{P}_{\theta}}\right] + \delta^{\alpha\beta}\left[\frac{\partial\rho U_m}{\partial\hat{\varepsilon}_{\lambda\theta}} + \hat{P}^{\lambda}\frac{\partial\rho U_m}{\partial\hat{P}_{\theta}}\right]\right)_0.$$

Here the notation $(\ )_0$ means that the quantity inside the parentheses is calculated for the initial state. The stress tensor components to within first-order infinitesimals are

$$P^{\alpha\beta}_{(m)} = P^{\alpha\beta}_{0} + \lambda^{\alpha\beta\lambda\theta}u_{\lambda\theta} + \zeta^{\alpha\beta\lambda}\pi_{\lambda} + \eta^{\alpha\beta}s_1 + (1/2)\, (E^{\lambda}_{0}\pi_{\lambda} - P^{\lambda}_{0}e_{\lambda})\, \delta^{\alpha\beta}$$

$$+ \tau^{\alpha\beta\lambda\theta}\, \partial\varepsilon_{\lambda\theta}/\partial t + \delta^{\alpha\beta}P^{0}_{\lambda}\Pi^{\lambda}_{1} + (1/2)\, (P^{\alpha}_{0}\Pi^{\beta}_{1} - P^{\beta}_{0}\Pi^{\alpha}_{1}),$$

where $P^{\alpha\beta}_{0}$ are the stress tensor components in the initial state:

$$P^{\alpha\beta}_{0} = (\hat{P}^{\alpha}\; \partial\rho U_m/\partial\hat{P}_{\beta} + \partial\rho U_m/\partial\hat{\varepsilon}_{\alpha\beta})_0 + \delta^{\alpha\beta}\, [\rho U_m - (1/2)\, P^{\lambda}E_{\lambda}]_0.$$

The total tensor of strain is defined by components

$$P^{\alpha\beta} = P^{\alpha\beta}_{(m)} + \frac{1}{4\pi}\left(D^{\beta}E^{\alpha} - \frac{1}{2}\delta^{\alpha\beta}E_{\lambda}D^{\lambda}\right) = P^{\alpha\beta}_{(0)} + \frac{1}{4\pi}\left(D^{\beta}_{0}E^{\alpha}_{0} - \frac{1}{2}\delta^{\alpha\beta}E^{0}_{\lambda}D^{\lambda}_{0}\right)$$

$$+ \sigma^{\alpha\beta} + N^{\alpha\beta}.$$

Here

$$\sigma^{\alpha\beta} = \lambda^{\alpha\beta\lambda\theta}u_{\lambda\theta} + \zeta^{\alpha\beta\lambda}\pi_{\lambda} + \eta^{\alpha\beta}s_1 + \tau^{\alpha\beta\lambda\theta}\partial\varepsilon_{\lambda\theta}/\partial t + \delta^{\alpha\beta}P^{0}_{\lambda}\Pi^{\lambda}_{1} + (1/2)\, (P^{\alpha}_{0}\Pi^{\beta}_{1}$$

$$- P^{\beta}_{0}\Pi^{\alpha}_{1}), \quad N^{\alpha\beta} = (1/4\pi)\, (E^{\alpha}_{0}\delta^{\beta} + D^{\beta}_{0}\, e^{\alpha} - \delta^{\alpha\beta}\, D^{0}_{\lambda}\, e^{\lambda}), \quad \delta^{\beta} = e^{\beta} + 4\pi\, \pi^{\beta}.$$

Obviously, $\partial_\beta N^{\alpha\beta} = 0$. We note that the tensor components $\zeta^{\mu\nu\alpha}$ in Eqs. (5.3.11) are generally asymmetric in the indices μ and ν. For this reason the polarization equation in (5.3.11) links the components e^α of the electric field vector not only with the strain tensor components $\varepsilon_{\alpha\beta} = (u_{\alpha\beta} + u_{\beta\alpha})/2$, as in known linear theories, but also with the components $\theta_\alpha = (1/2)\,\mathrm{curl}_\alpha\theta$ of the vector representing the rotation of the strain axes. In the case of axial symmetry (e.g. polarized ceramics), the coefficients b, f, η, ζ, and λ in Eqs. (5.3.11) may be defined via the following relationships:

$$\begin{aligned}
&b^{\alpha\beta} = b_1\,\delta^{\alpha\beta} + b_2 n^\alpha n^\beta, \quad f^\alpha = f n^\alpha, \quad \eta^{\alpha\beta} = \eta_1\delta^{\alpha\beta} + \eta_2 n^\alpha n^\beta,\\
&\zeta^{\alpha\beta\lambda} = \zeta_1 n^\alpha n^\beta n^\lambda + \zeta_2\delta^{\alpha\beta}n^\lambda + \zeta_3\delta^{\beta\lambda}n^\alpha + \zeta_4\delta^{\alpha\lambda}n^\beta,\\
&\lambda^{\alpha\beta\lambda\theta} = \lambda_1\delta^{\alpha\beta}\delta^{\lambda\theta} + \lambda_2\,(\delta^{\alpha\lambda}\delta^{\beta\theta} + \delta^{\alpha\theta}\delta^{\beta\lambda}) + \lambda_3\delta^{\alpha\lambda}n^\beta n^\theta\\
&+ \lambda_4\,(\delta^{\alpha\theta}n^\beta n^\lambda + \delta^{\lambda\beta}n^\alpha n^\theta) + \lambda_5\delta^{\beta\theta}n^\alpha n^\lambda\\
&+ \lambda_6\,(\delta^{\alpha\beta}n^\lambda n^\theta + \delta^{\lambda\theta}n^\alpha n^\beta) + \lambda_7 n^\alpha n^\beta n^\lambda n^\theta,
\end{aligned}$$

where n^α are the components of a unit vector directed along the anisotropy axis. Generally, these coefficients are interdependent.

5.4 MODELS OF MAGNETIZABLE LIQUIDS WITH INTRINSIC MOMENT OF MOMENTUM

Let us study the class of models of magnetizable liquids with intrinsic angular momentum, a class determined by the function Λ_0

$$\begin{aligned}
\Lambda_0 = &-\rho U_m\,(\rho,\ s,\ m_\alpha,\ e_{\alpha\beta},\ K_B) + (1/g)\,\rho m^\alpha\Omega_\alpha\\
&- \gamma\rho m^\alpha\omega_\alpha + (1/2)\,\rho I\Omega^2,
\end{aligned} \tag{5.4.1}$$

with I, g, and γ constants. We write the dynamical equations corresponding to the function (5.4.1) assuming that Π^α is zero and $\tau_\alpha{}^\beta$, $A^{\alpha\beta}$, $\overset{*}{R}{}^\alpha$, and L^α are defined in (5.2.21) with $l^\alpha_\beta = \delta^\alpha_\beta$.

$$\begin{aligned}
&B^\alpha - \frac{1}{c}[\mathbf{v},\ \mathbf{E}]^\alpha = \frac{\partial U_m}{\partial m_\alpha} + \gamma\omega^\alpha - \frac{1}{g}\Omega^\alpha + \eta^{\alpha\beta}\rho\left(\frac{dm_\beta}{dt} - [\boldsymbol{\omega},\ \mathbf{m}]_\beta\right),\\
&\rho\frac{d}{dt}\left(\frac{1}{g}\ m^\alpha + I\Omega^\alpha\right) - \partial_\beta\,(a^{\alpha\beta\lambda\theta}\partial_\lambda\Omega_\theta) + \frac{1}{g}[\rho\mathbf{m},\ \boldsymbol{\Omega}]^\alpha = -r^{\alpha\beta}\,(\Omega_\beta - \omega_\beta),\\
&T = \partial U_m/\partial s, \quad \partial\rho/\partial t + \mathrm{div}\,\rho\mathbf{v} = 0, \quad \partial\rho_e/\partial t + \mathrm{div}\,(\rho_e\mathbf{v} + \mathbf{i}) = 0,\\
&\rho T\,ds/dt = -\partial_\alpha q^\alpha + \tau^{\alpha\beta\lambda\theta}e_{\alpha\beta}e_{\lambda\theta} + a^{\alpha\beta\lambda\theta}\,\partial_\beta\Omega_\alpha\,\partial_\lambda\Omega_\theta\\
&+ \sigma^{\alpha\beta}\,(E_\beta + (1/c)\,[\mathbf{v},\ \mathbf{B}]_\beta)\,(E_\alpha + (1/c)\,[\mathbf{v},\ \mathbf{B}]_\alpha) + r^{\alpha\beta}\,(\Omega_\alpha\\
&- \omega_\alpha)\,(\Omega_\beta - \omega_\beta) + \rho^2\eta^{\alpha\beta}\,(dm_\alpha/dt - [\boldsymbol{\omega},\ \mathbf{m}]_\alpha)\,(dm_\beta/dt - [\boldsymbol{\omega},\ \mathbf{m}]_\beta).
\end{aligned} \tag{5.4.2}$$

To make the set of equations (5.4.2) closed, we must add the Maxwell equations (5.2.1) and the momentum equations (5.2.7), in which the components of the momentum density vector and of the stress tensor are

$$g^{\alpha}_{(m)} = \rho v^{\alpha} - (1/2)\,\gamma\,\mathrm{curl}^{\alpha}\,\rho\mathbf{m} + \partial_{\beta}\,(\partial\rho U_m/\partial e_{\alpha\beta}),$$

$$\begin{aligned} P_{\alpha}{}^{\beta}_{(m)} = {} & -p\delta^{\beta}_{\alpha} + (1/2)\,\gamma\varepsilon^{\beta\lambda\mu}\,\rho m_{\lambda}\partial_{\alpha}v_{\mu} - \rho\,(\partial U_m/\partial e_{\beta\lambda})\,\partial_{\alpha}v_{\lambda} \\ & + \tau_{\alpha}{}^{\beta\lambda\theta}\,e_{\lambda\theta} - (1/2)\,\varepsilon_{\alpha}{}^{\beta\lambda}\overset{*}{R}_{\lambda} \\ & + (1/2)\,\rho\,(m_{\alpha}L^{\beta} - m^{\beta}L_{\alpha}), \end{aligned} \tag{5.4.3}$$

where the pressure p is

$$p = \rho^2\,\partial U_m/\partial\rho - (1/2)\,\rho m_{\alpha}\,(B^{\alpha} - (1/c)\,[\mathbf{v},\ \mathbf{E}]^{\alpha}). \tag{5.4.4}$$

According to definitions (5.2.6), within the scope of Eqs. (5.4.2) the couple stress tensor is given only by its components $A^{\alpha\beta}$, while the intrinsic moment of momentum vector is given by the sum of a term proportional to the magnetization vector and a term proportional to the intrinsic rotation vector, i.e.

$$k^{\alpha} = (1/g)\,m^{\alpha} + I\Omega^{\alpha}. \tag{5.4.5}$$

If $1/g$ is nonzero, the first equation in (5.4.2) enables us to drop the internal rotation vector from the other equations. For one, the angular momentum equation (the second equation in (5.4.2)) becomes, after we drop $\mathbf{\Omega}$,

$$\begin{aligned} & \rho\,\frac{d}{dt}\left(\frac{1}{g}\,m^{\alpha} - Ig\overset{*}{H}{}^{\alpha}\right) + g\partial_{\beta}\,(a^{\alpha\beta\lambda\theta}\partial_{\lambda}\overset{*}{H}_{\theta}) \\ & = [\rho\mathbf{m},\ \overset{*}{\mathbf{H}}]^{\alpha} + gr^{\alpha\beta}\left(\overset{*}{H}_{\beta} + \frac{1}{g}\,\omega_{\beta}\right), \end{aligned} \tag{5.4.6}$$

where $\overset{*}{H}{}^{\alpha}$ are the components of the effective magnetic field:

$$\begin{aligned} \overset{*}{H}{}^{\alpha} = {} & B^{\alpha} - \frac{1}{c}\,[\mathbf{v},\ \mathbf{E}]^{\alpha} - \frac{\partial U_m}{\partial m_{\alpha}} \\ & - \gamma\omega^{\alpha} - \eta^{\alpha\beta}\rho\left(\frac{dm_{\beta}}{dt} - [\boldsymbol{\omega},\ \mathbf{m}]_{\beta}\right). \end{aligned} \tag{5.4.7}$$

Equations (5.4.2) can be used, for instance, to describe ferromagnetic liquids, which constitute a suspension of ferromagnetic particles in a liquid carrier. Below we consider some problems that are within the scope of Eqs. (5.4.2) in the magnetostatic approximation.

When considering specific physical models and processes, we can simplify Eqs. (5.4.2). For instance, allowance for intrinsic moment of momentum for ferromagnetic liquids is expedient, generally speaking, when high-frequency processes are studied. They can be neglected in many hydromechanical problems.

5.5 COUETTE FLOW OF AN INCOMPRESSIBLE VISCOUS MAGNETIZABLE LIQUID

Let us consider the model of an isotropic liquid for which the function U_m in (5.4.1) depends solely on the invariant $M = (M_\alpha M^\alpha)^{1/2}$ and $i^\beta = \rho_e = 0$, while the $\tau^{\alpha\beta\lambda\theta}$, $r^{\alpha\beta}$, $\eta^{\alpha\beta}$, and $a^{\alpha\beta\lambda\theta}$ in Eqs. (5.2.21) are defined by the following equalities:

$$\begin{aligned}
&\eta^{\alpha\beta} = \theta\,(\delta^{\alpha\beta} - n^\alpha n^\beta) + \theta_{\|} n^\alpha n^\beta,\\
&r^{\alpha\beta} = (1/g^2)\,[(1/\tau)\,(\delta^{\alpha\beta} - n^\alpha n^\beta) + (1/\tau_{\|})\,n^\alpha n^\beta],\\
&a_{\alpha\beta\lambda\theta} = a_1\delta^{\alpha\beta}\delta^{\lambda\theta} + a_2\delta^{\alpha\lambda}\delta^{\beta\theta} + a_3\delta^{\alpha\theta}\delta^{\beta\lambda},\\
&\tau^{\alpha\beta\lambda\theta} = \mu\,(\delta^{\alpha\lambda}\delta^{\beta\theta} + \delta^{\alpha\theta}\delta^{\beta\lambda}),
\end{aligned}\tag{5.5.1}$$

where μ is the viscosity coefficient, θ and τ relaxation times, and $n^\alpha = M^\alpha/M$ the components of a unit vector directed along the magnetization vector. We assume all coefficients θ, τ, a, and μ to be constant.

The definitions (5.5.1) enable us to write a closed set of equations describing, in the magnetostatic approximation, the flow of an incompressible viscous liquid in the absence of external forces:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{g}M^\alpha - \rho I g\overset{*}{H}{}^\alpha\right) + g a_3\,\Delta\overset{*}{H}{}^\alpha + g\,(a_1 + a_2)\,\partial^\alpha\partial_\beta\overset{*}{H}{}^\beta\\
&= [\mathbf{M},\ \overset{*}{\mathbf{H}}]^\alpha + (1/g)\,[(1/\tau)\,(\delta^{\alpha\beta} - n^\alpha n^\beta)\\
&+ (1/\tau_{\|})\,n^\alpha n^\beta]\,[\overset{*}{H}_\beta + (1/g)\,\omega_\beta],\\
&\operatorname{div}\mathbf{v} = 0,\ \ \operatorname{curl}\mathbf{H} = 0,\ \ \operatorname{div}(\mathbf{H} + 4\pi\mathbf{M}) = 0,\\
&\frac{d}{dt}\left(\rho v^\alpha - \frac{1}{2}\gamma\,\mathrm{curl}^\alpha\mathbf{M}\right) = -\partial^\alpha\overset{*}{p} + \mu\Delta v^\alpha\\
&+ (1/2)\,\gamma\,(\mathrm{curl}^\lambda\mathbf{M})\,\partial^\alpha v_\lambda + \gamma\omega^\lambda\partial^\alpha M_\lambda + M^\lambda\partial^\alpha H_\lambda\\
&+ (1/2)\,\partial_\beta\,(M^\alpha L^\beta - M^\beta L^\alpha - \varepsilon^{\alpha\beta\lambda}\overset{*}{R}_\lambda),\\
&L^\alpha + [\theta\,(\delta^{\alpha\beta} - n^\alpha n^\beta) + \theta_{\|}\,n^\alpha n^\beta]\,(dM_\beta/dt - [\omega,\ \mathbf{M}]_\beta),\\
&\overset{*}{R}{}^\alpha = (1/g)\,[(1/\tau)\,(\delta^{\alpha\beta} - n^\alpha n^\beta) + (1/\tau_{\|})\,n^\alpha n^\beta]\,[\overset{*}{H}_\beta + (1/g)\,\omega_\beta],\\
&\overset{*}{H}{}^\alpha = H^\alpha - (1/\chi)\,M^\alpha - \gamma\omega^\alpha - [\theta\,(\delta^{\alpha\beta} - n^\alpha n^\beta)\\
&+ \theta_{\|} n^\alpha n^\beta\,(dM_\beta/dt - [\omega,\ \mathbf{M}]_\beta),\\
&1/\chi = -4\pi + (\rho/M)\,(\partial U_m/\partial M),
\end{aligned}\tag{5.5.2}$$

where $\overset{*}{p}$ is a function related to pressure.

Let us assume that the observer's coordinate system with variables x, y, and z is inertial and Cartesian. Consider the steady flow of the liquid described by Eqs. (5.5.2) between two parallel planes $z = 0$ and $z = d$ in a given constant magnetic field B^0. The plane $z = 0$ is fixed while plane $z = d$ moves with a constant velocity $\mathbf{v} = (2\omega_0 d, 0, 0)$, with ω_0 a constant. Equations (5.5.2) and the appropriate (known) boundary conditions are valid if in the region $0 \leqslant z \leqslant d$ we put

$$\mathbf{v} = (2\omega_0 z, 0, 0), \quad \mathbf{H} = (B_1^0, B_2^0, B_3^0 - 4\pi M_3), \tag{5.5.3}$$

with parameters $\overset{*}{p}$ and M constant. The components M^α of the magnetization vector are expressed in terms of H and ω_0 via the first equation in (5.5.2). Assuming the dimensionless parameters $\omega_0\tau$ and $\omega_0\theta$ to be small (which is the case in real systems), we acquire an approximate formula

$$M^\alpha = \chi H^\alpha \left(1 + \frac{g\chi^2\tau^2}{1+\varkappa^2} H_\lambda \omega^\lambda\right) + \chi^2 \left(\theta + \frac{\tau}{1+\varkappa^2}\right) [\omega, \mathbf{H}]^\alpha$$
$$+ \frac{\chi}{g}\left(-g\gamma + \frac{1}{1+\varkappa^2}\right)\omega^\alpha, \tag{5.5.4}$$

where $\varkappa^2 = \chi^2 H^2 g^2 \tau^2$, and $\omega^\alpha = (0, \omega_0, 0)$ are the components of the vorticity (an exact solution for $\theta = 0$ is given in [71]). The components f^α of the force acting on plane $z = 0$ are

$$f^\alpha = \left[-\frac{1}{4\pi}\left(H^\alpha B^\beta - \frac{1}{2}\,\delta^{\alpha\beta} H_\lambda B^\lambda\right) + v^\beta g^\alpha_{(m)}\right.$$
$$\left. - P^{\alpha\beta}_{(m)} + \frac{1}{4\pi}\left(B_0^\alpha B_0^\beta - \frac{1}{2}\,B_0^2 \delta^{\alpha\beta}\right)\right] \nu_\beta, \tag{5.5.5}$$

where $\nu_\beta = (0, 0, -1)$ are the components of a unit vector normal to plane $z = 0$. Calculations that allow for (5.5.4) yield

$$f^x = 2\omega_0 \left[\mu + \frac{1}{4}\chi^2 H^2 \left(\theta + \frac{\tau}{1+\chi^2 H^2 g^2 \tau^2}\right)\sin^2\alpha\right], \tag{5.5.6}$$

where α is the angle between the magnetic field strength $\mathbf{H}$ and the vorticity $\boldsymbol{\omega}$. The expression in brackets in (5.5.6) is the total "effective" viscosity coefficient of the liquid, and we see that it depends on the strength of the field. If the magnetic field strength and the vorticity are parallel, the total viscosity coefficient is exactly μ.

5.6 POISEUILLE FLOW IN CYLINDRICAL CHANNEL

Let us consider the steady flow of a liquid described by Eqs. (5.5.2) with $a_1 = a_2 = a_3 = 0$, $\chi = \text{const}$ in a cylindrical channel with a circular cross section of radius R in an external constant field $\mathbf{B}^0$ directed along the channel's axis. The walls of the channel are made of a nonmagnetic material. We select a cylindrical system of coordinates r, φ, z attached to the channel and we ask for a solution of Eqs. (5.5.2) in the form

$$\mathbf{v} = (0, v_\varphi, v_z), \quad \mathbf{M} = (M_r, M_\varphi, M_z), \quad \mathbf{H} = (-4\pi M_r, 0, H), \tag{5.6.1}$$

with v and M depending only on r, and $H = B^0$ constant; the quantity $\overset{*}{p}$ in the momentum equation may depend on r and z.

Under these assumptions the Maxwell equations and the boundary conditions for the field are satisfied everywhere, while the momentum equations in (5.5.2) admit of two integrals of the motion:

$$\begin{gathered}
\mu\left(\frac{dv_\varphi}{dr} - \frac{1}{r} v_\varphi\right) - \left(2\pi M_\varphi + \frac{1}{4}\gamma \frac{dv_z}{dr}\right) M_r = \frac{C_1}{r^2}, \\
\mu \frac{dv_z}{dr} + \frac{1}{2}\left[M_z H_r - M_r\left(H_z - \frac{\gamma}{2}\frac{dv_\varphi}{dr} + \frac{2 - g\gamma}{2g}\frac{v_\varphi}{r}\right)\right] \\
= -\frac{2\mu^2}{\rho R^3} r\varepsilon + \frac{C_2}{r},
\end{gathered} \tag{5.6.2}$$

with

$$\varepsilon = -(\rho R^3/4\mu^2)\, \partial \overset{*}{p}/\partial z, \tag{5.6.3}$$

and C_1 and C_2 arbitrary constants. The projection of the momentum equation on the r-axis determines the dependence of $\overset{*}{p}$ on r.

The moment of momentum equations in (5.5.2) can be written thus:

$$\begin{gathered}
M_\varphi\left[gH + \frac{g\theta}{2}\frac{dv_z}{dr} - \frac{g\gamma\tau + \theta}{2\tau}\frac{dv_\varphi}{dr}\right. \\
\left. + \left(1 + \frac{\theta - g\gamma\tau}{2\tau}\right)\frac{v_\varphi}{r}\right] - \frac{1 + 4\pi\chi}{\tau\chi} M_r \\
- M_z\left[\frac{\theta + g\gamma\tau}{2\tau}\frac{dv_z}{dr} + \frac{g\theta}{2} M_r\left(\frac{dv_\varphi}{dr} - \frac{v_\varphi}{r}\right)\right] = 0, \\
\frac{1}{\tau\chi} M_\varphi + M_r\left[gH + \frac{g\theta}{2} M_r \frac{dv_z}{dr} - \frac{\theta + g\gamma\tau}{2\tau}\frac{dv_\varphi}{dr}\right. \\
\left. + \left(1 + \frac{\theta - g\gamma\tau}{2\tau}\right)\frac{v_\varphi}{r}\right] + M_z\left[4\pi g M_r\right. \\
\left. + \frac{g\theta}{2} M_z \frac{dv_z}{dr} + \frac{g\theta}{2} M_\varphi\left(\frac{dv_\varphi}{dr} - \frac{v_\varphi}{r}\right)\right] - \frac{g\gamma - 1}{2g\tau}\frac{dv_z}{dr} = 0. \\
M_\varphi\left[4\pi g M_r + \frac{g\theta}{2} M_z \frac{dv_z}{dr} + \frac{g\theta}{2} M_\varphi\left(\frac{dv_\varphi}{dr} - \frac{v_\varphi}{r}\right)\right] \\
+ M_r\left[\frac{g\theta}{2} M_r\left(\frac{dv_\varphi}{dr} - \frac{v_\varphi}{r}\right) + \frac{\theta + g\gamma\tau}{2\tau}\frac{dv_z}{dr}\right] \\
+ \frac{1}{\tau}\left(H - \frac{1}{\chi} M_z\right) + \frac{1 - g\gamma}{2g\tau}\left(\frac{dv_\varphi}{dr} + \frac{v_\varphi}{r}\right) = 0.
\end{gathered} \tag{5.6.4}$$

Consider the case where the dimensionless parameter determining the drop in pressure in the channel is small. Confining ourselves to second-order terms, we put

$$v = \overset{0}{v} + \overset{1}{v}\varepsilon + \overset{2}{v}\varepsilon^2, \quad M = \overset{0}{M} + \overset{1}{M}\varepsilon + \overset{2}{M}\varepsilon^2. \tag{5.6.5}$$

Then Eqs. (5.6.2) and (5.6.4) yield the following equations for the functions $\overset{1}{v}$, $\overset{2}{v}$, $\overset{1}{M}$, and $\overset{2}{M}$:

$$\begin{gathered}
\overset{0}{M}_\varphi = \overset{0}{M}_r = \overset{2}{M}_\varphi = \overset{2}{M}_r = \overset{1}{M}_z = 0, \quad \overset{0}{M}_z = \chi H, \\
\overset{0}{v}_\varphi = \overset{0}{v}_z = \overset{1}{v}_\varphi = \overset{2}{v}_z = 0, \\
gH\overset{1}{M}_\varphi - \frac{1+4\pi\chi}{\tau\chi}\overset{1}{M}_r - \overset{0}{M}_z \frac{\theta + g\gamma\tau}{2\tau}\frac{d\overset{1}{v}_z}{dr} = 0, \\
\frac{1}{\tau\chi}\overset{1}{M}_\varphi + g(H + 4\pi\overset{0}{M}_z)\overset{1}{M}_r - \frac{1}{2}\left(\frac{g\gamma - 1}{g\tau} - g\theta\overset{0}{M}{}_z^2\right)\frac{d\overset{1}{v}_z}{dr} = 0, \\
\mu\frac{d\overset{1}{v}_z}{dr} - \frac{1}{2}(H + 4\pi\overset{0}{M}_z)\overset{1}{M}_r = -\frac{2\mu^2}{\rho R^3} r, \\
\mu\left(\frac{d\overset{2}{v}_\varphi}{dr} - \frac{1}{r}\overset{2}{v}_\varphi\right) - \left(2\pi\overset{1}{M}_\varphi + \frac{1}{4}\gamma\frac{d\overset{1}{v}_z}{dr}\right)\overset{1}{M}_r = 0, \\
4\pi g\overset{1}{M}_\varphi\overset{1}{M}_r + \frac{1}{2}g\theta\overset{0}{M}_z\overset{1}{M}_\varphi\frac{d\overset{1}{v}_z}{dr} + \frac{\theta + g\gamma\tau}{2\tau}\overset{1}{M}_r\frac{d\overset{1}{v}_z}{dr} - \frac{1}{\tau\chi}\overset{2}{M}_z \\
+ \frac{1 - g\gamma}{2g\tau}\left(\frac{d\overset{2}{v}_\varphi}{dr} + \frac{1}{r}\overset{2}{v}_\varphi\right) = 0.
\end{gathered} \tag{5.6.6}$$

Equations (5.6.5) and (5.6.6) have the following solution that satisfies the condition for the zero velocity at the wall of the channel:

$$\begin{gathered}
M_r = \frac{1}{H_0(1 + 4\pi\chi)}\frac{\mu - \mu_e}{\mu_e} r \frac{\partial \overset{*}{p}}{\partial z}, \\
M_\varphi = \frac{\chi}{4g\mu_e}\left(g\gamma - \frac{1}{1 + \varkappa^2}\right) r \frac{\partial \overset{*}{p}}{\partial z}, \\
v_z = \frac{1}{4\mu_e}(r^2 - R^2)\frac{\partial \overset{*}{p}}{\partial z}, \\
v_\varphi = r(r^2 - R^2)\left(\frac{\partial \overset{*}{p}}{\partial z}\right)^2 \frac{\mu_e - \mu}{16\mu\mu_e^2 gH}\left(-g\gamma + \frac{4\pi\chi}{1 + 4\pi\chi}\frac{1}{1 + \varkappa^2}\right),
\end{gathered} \tag{5.6.7}$$

where μ_e is the effective viscosity coefficient (the same as in the Couette problem in (5.5.6), since within the same precision $\alpha = \pi/2$):

$$\mu_e = \mu + \frac{1}{4}\chi^2 H^2\left(\theta + \frac{\tau}{1 + \chi^2 H^2 g^2 \tau^2}\right).$$

The solution shows that with a drop in pressure a magnetized liquid in the channel undergoes helical motion. For the discharge Q we have the usual relationship, in which we must substitute the effective viscosity coefficient μ_e for the ordinary, μ:

$$Q = -\frac{\pi R^4}{8\mu_e}\frac{\partial \overset{*}{p}}{\partial z}.$$

5.7 MAGNETOACOUSTIC WAVES IN MAGNETIZABLE LIQUIDS

Let us consider, in the magnetostatic approximation, the isentropic flow of an uncharged, nonconducting liquid described by Eqs. (5.4.2) and (5.4.3), with Λ_0 in the following form:

$$\Lambda_0 = -\rho U_m(\rho, M) - 2\pi M^2 + (1/g) M^\alpha \Omega_\alpha + (1/2)\rho I \Omega_\alpha \Omega^\alpha - \gamma M^\alpha \omega_\alpha - (1/2)\nu_1 M^\alpha M^\beta e_{\alpha\beta} - (1/2)\nu_2 M^2 e_\alpha{}^\alpha, \tag{5.7.1}$$

where $M = (M_\alpha M^\alpha)^{1/2}$, and ν_1 and ν_2 are constants. The coefficients $r^{\alpha\beta}$ and $\eta^{\alpha\beta}$ are defined by (5.5.1), and for $a^{\alpha\beta\lambda\theta}$ and $\tau^{\alpha\beta\lambda\theta}$ we take

$$a^{\alpha\beta\lambda\theta} = (D/g^2)\,\delta^{\alpha\theta}\delta^{\beta\lambda}, \quad \tau^{\alpha\beta\lambda\theta} = \mu\,(\delta^{\alpha\lambda}\delta^{\beta\theta} + \delta^{\alpha\theta}\delta^{\beta\lambda}) + \lambda\delta^{\alpha\beta}\delta^{\lambda\theta},$$

with μ and λ the viscosity coefficients, and D a constant.

Suppose that M_0, H_0, ρ_0, and $v_0 = 0$ are constant values of the parameters of the liquid in equilibrium. If we assume that the functions $M(x^\alpha, t)$, $H(x^\alpha, t)$, $\rho(x^\alpha, t)$, and $v(x^\alpha, t)$ vary little with respect to the equilibrium values

$$M = M_0 + \mu, \quad H = H_0 + h, \quad \rho = \rho_0 + \rho_1,$$

then to within second-order terms we have

$$\rho U_m = \frac{1}{2\rho_0} a_0^2 \rho_1^2 + \frac{1}{2}\beta^{\alpha\beta}\mu_\alpha\mu_\beta + b n^\alpha \mu_\alpha \rho_1 + \beta_1 n^\alpha \mu_\alpha + A\rho_1 + \text{const.} \tag{5.7.2}$$

Here

$$n^\alpha = M_0^\alpha / M_0, \quad \beta^{\alpha\beta} = \beta_1 \delta^{\alpha\beta} + \beta_2 n^\alpha n^\beta,$$
$$\beta_1 = (\rho_0/M_0)(\partial U_m/\partial M)_0,$$
$$\beta_2 = \rho_0\left(\frac{\partial^2 U_m}{\partial M^2}\right)_0 - \frac{\rho_0}{M_0}\left(\frac{\partial U_m}{\partial M}\right)_0,$$
$$b = \left(\frac{\partial U_m}{\partial M}\right)_0 + \rho_0\left(\frac{\partial^2 U_m}{\partial \rho\, \partial M}\right)_0, \tag{5.7.3}$$
$$a_0^2 = \rho_0^2\left(\frac{\partial^2 U_m}{\partial \rho^2}\right)_0 + 2\rho_0\left(\frac{\partial U_m}{\partial \rho^2}\right)_0, \quad A = \left(U_m + \rho\frac{\partial U_m}{\partial \rho}\right)_0.$$

The symbol $(\ )_0$ means that the expressions in the parentheses are taken at equilibrium values of the parameters.

Taking into account (5.7.1) and (5.7.2), we linearize Eqs. (5.4.2) and (5.4.6) in the magnetostatic approximation for the functions M, H, ρ, and v corresponding to Λ_0 defined in (5.7.1). This yields

$$\begin{gathered}
\frac{\partial}{\partial t}(\mu^\alpha-\rho_0 I g^2 \overset{*}{h}{}^\alpha)+D\Delta\overset{*}{h}{}^\alpha+M_0^\alpha\partial_\lambda v^\lambda=g\,[\mathbf{M}_0,\overset{*}{\mathbf{h}}]^\alpha+g^2 r^{\alpha\beta}\left(\overset{*}{h}_\beta+\frac{1}{g}\omega_\beta\right),\\
\frac{\partial}{\partial t}\left\{\rho_0 v^\alpha-\frac{1}{2}\gamma\ \mathrm{curl}^\alpha\,\mu+\partial_\lambda\left[\frac{1}{2}\nu_1(M_0^\lambda\mu^\alpha+M_0^\alpha\mu^\lambda)\right.\right.\\
\left.\left.+\nu_2 M_0^\beta\mu_\beta\delta^{\alpha\lambda}\right]\right\}=-\partial^\alpha\tilde{p}+\mu\Delta v^\alpha+(\lambda+\mu)\,\partial^\alpha\partial_\beta v^\beta\\
+M_0^\lambda\partial^\alpha h_\lambda+\frac{1}{2}\partial_\beta\left[\theta M_0^\alpha\left(\frac{\partial\mu^\beta}{\partial t}-[\omega,\mathbf{M}_0]^\beta\right)\right.\\
\left.-\theta M_0^\beta\left(\frac{\partial\mu^\alpha}{\partial t}-[\omega,\mathbf{M}_0]^\alpha\right)-\frac{1}{g}\varepsilon^{\alpha\beta\lambda}r_{\lambda\theta}\left(\overset{*}{h}{}^\theta+\frac{1}{g}\omega^\theta\right)\right],\\
\mathrm{curl}\ \mathbf{h}=0,\quad \mathrm{div}\,(\mathbf{h}+4\pi\boldsymbol{\mu})=0,\quad \frac{\partial\rho_1}{\partial t}+\rho_0\,\mathrm{div}\,\mathbf{v}=0.
\end{gathered}\tag{5.7.4}$$

Here

$$\begin{gathered}
\overset{*}{h}{}^\alpha=h^\alpha-\beta^{\alpha\beta}\mu_\beta-\gamma\omega^\alpha-\nu_1 M_\lambda^0 e^{\alpha\lambda}-\nu_2 M_0^\alpha\partial_\lambda v^\lambda\\
-bn^\alpha\rho_1-[\theta(\delta^{\alpha\lambda}-n^\alpha n^\lambda)+\theta_{||}n^\alpha n^\lambda]\\
\times\left(\frac{\partial\mu_\lambda}{\partial t}-[\omega,\mathbf{M}_0]_\lambda+M_\lambda^0\partial_\beta v^\beta\right),
\end{gathered}\tag{5.7.5}$$

$$\begin{gathered}
\tilde{p}=(a_0^2+bM_0)\,\rho_1+[\rho_0 b+M_0(\beta_1+\beta_2)]\,n^\alpha\mu_\alpha+\gamma M_0^\alpha\omega_\alpha\\
+\nu_1 M_0^\alpha M_0^\lambda e_{\alpha\lambda}+\nu_2 M_0^2\,\partial_\alpha v^\alpha.
\end{gathered}$$

5.7.1. Spin Waves. Let us consider spin waves in a magnetizable liquid in the linear approximation, i.e. small adiabatic oscillations in the liquid's magnetization and negligible acoustic waves. In accordance with the approximation, the initial equations can be written in the following manner:

$$\begin{gathered}
\frac{\partial}{\partial t}(\mu^\alpha-\rho_0 I g^2\overset{*}{h}{}^\alpha)+D\Delta\overset{*}{h}{}^\alpha=g\,[\mathbf{M}_0,\overset{*}{\mathbf{h}}]^\alpha+\\
+\left[\frac{1}{\tau}(\delta^{\alpha\beta}-n^\alpha n^\beta)+\frac{1}{\tau_{||}}n^\alpha n^\beta\right]\overset{*}{h}_\beta,\\
\mathrm{curl}\ \mathbf{h}=0,\quad \mathrm{div}\,(\mathbf{h}+4\pi\boldsymbol{\mu})=0,
\end{gathered}\tag{5.7.6}$$

$$\overset{*}{h}{}^\alpha=h^\alpha-\beta^{\alpha\beta}\mu_\beta-[\theta(\delta^{\alpha\lambda}-n^\alpha n^\lambda)+\theta_{||}n^\alpha n^\lambda]\,\partial\mu_\lambda/\partial t.$$

Assuming that $\mu=\mu_*\exp[i(-\omega t+k_\alpha x^\alpha)]$ and $h^\alpha=h_*^\alpha\exp[i(-\omega t+k_\alpha x^\alpha)]$, with ω the frequency of the waves and k^α the wave vector components, from Eqs. (5.7.6) we find that

$$\mu^\alpha=\chi^{\alpha\beta}h_\beta,\quad h^\alpha=-(4\pi/k^2)\,k^\alpha k_\beta\mu^\beta,\tag{5.7.7}$$

where the components $\chi^{\alpha\beta}$ of the high-frequency magnetic susceptibility tensor are given by the relationship

$$\chi^{\alpha\beta} = \chi_1\delta^{\alpha\beta} + \chi_2 n^\alpha n^\beta + i\chi_3 n_\lambda \varepsilon^{\alpha\beta\lambda}, \tag{5.7.8}$$

with χ_1, χ_2, and χ_3 scalar coefficients calculated via the following formulas:

$$\begin{aligned}
\chi_1 &= \frac{1/2}{\beta_1 - i\omega\theta + \omega\,(i/\tau - iDk^2 - gM_0 + \omega\rho_0 I g^2)^{-1}} \\
&\quad + \frac{1/2}{\beta_1 - i\omega\theta + \omega\,(i/\tau - iDk^2 + gM_0 + \omega\rho_0 I g^2)^{-1}}, \\
\chi_3 &= \frac{1/2}{\beta_1 - i\omega\theta + \omega\,(i/\tau - iDk^2 - gM_0 + \omega\rho_0 I g^2)^{-1}} \\
&\quad - \frac{1/2}{\beta_1 - i\omega\theta + \omega\,(i/\tau - iDk^2 + gM_0 + \omega\rho_0 I g^2)^{-1}}, \\
\chi_2 &= -\chi_1 + \frac{1}{\beta_1 + \beta_2 - i\omega\theta_{\parallel} + \omega\,(i/\tau_{\parallel} - iDk^2 + \omega\rho_0 I g^2)^{-1}}.
\end{aligned} \tag{5.7.9}$$

From Eqs. (5.7.6) we obtain the dispersion equation for spin waves

$$k^2 + 4\pi k^\alpha k^\beta \chi_{\alpha\beta} = 0, \quad \text{or} \quad k^2 + 4\pi\,[k^2\chi_1 + (k_\alpha n^\alpha)^2\,\chi_2] = 0.$$

In the approximation with zero energy dissipation (i.e. $\theta = \theta_{\parallel} = D = \tau^{-1} = \tau_{\parallel}^{-1} = 0$), the dispersion equation yields an expression for the frequency of spin waves that depends only on the direction of the wave vector and not on its length:

$$\omega = \pm\frac{gM_0}{1 + \rho_0 I g^2 \beta_1}\left\{\beta_1^2 + \frac{4\pi\beta_1 \sin^2\psi}{1 + \rho_0 I g^2(4\pi + \beta_1) - 4\pi\rho_0^2 I^2 g^4(\cos^2\psi)\beta_2/[1 + \rho_0 I g^2(\beta_1 + \beta_2)]}\right\}^{1/2};$$

here ψ is the angle between wave vector **k** and vector $\mathbf{M}_0$.

Now we take a finite volume of the magnetizable liquid restricted by an ellipsoidal surface in an external magnetic field that changes with time according to the law $\exp(-i\omega t)$. We assume that the wavelength of the external field is much longer than the characteristic dimension l of the liquid ellipsoid, so that we can assume the field to be homogeneous over distances of order l. In this case a homogeneous magnetic field is generated in the liquid, and the frequency of the magnetic resonance appearing as a result is determined from the equation

$$\det(1 + 4\pi N\chi) = 0, \tag{5.7.10}$$

where 1 is a unit 3×3 matrix, N the matrix of the components of the tensor of demagnetizing coefficients, and χ the matrix of the components of the high-frequency magnetic susceptibility tensor calculated in the absence of dissipation: $\theta = \theta_{\parallel} = D = \tau^{-1} = \tau_{\parallel} = 0$.

Directing the major axes of the liquid ellipsoid along the coordinate axes, i.e.

$$N = \begin{Vmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{Vmatrix},$$

from Eq. (5.7.10) we find expression for the resonance frequency ω_*:

$$\omega_* = \left(\frac{g^2 M_0^2 (\beta_1 + 4\pi N_1)(\beta_1 + 4\pi N_2)}{[1 + \rho_0 I g^2 (\beta_1 + 4\pi N_1)][1 + \rho_0 I g^2 (\beta_1 + 4\pi N_2)]} \right)^{1/2}.$$

If $N_1 = N_2 = N$ (e.g. $N = 1/3$ for a sphere and $N = 1/2$ for a cylinder), the formula for ω_* simplifies:

$$\omega_* = \frac{g M_0 (\beta_1 + 4\pi N)}{1 + \rho_0 I g^2 (\beta_1 + 4\pi N)}.$$

Thus, the frequency of a homogeneous magnetic resonance depends on the coefficient I, which determines the energy of intrinsic rotation. Estimates of ω_* for ferromagnetic liquids show that allowing for the energy of intrinsic rotation may considerably decrease the resonance frequency.

5.7.2. Magnetoacoustic Waves. Take Eqs. (5.7.4) in the approximation with zero energy dissipation, i.e. $\eta^{\alpha\beta} = r^{\alpha\beta} = D = \lambda = \mu = 0$. We ask for a solution of these equations in the form of a harmonic wave propagating along the vector of constant magnetization of the liquid $k^\alpha = kn^\alpha$. Then Eqs. (5.7.4) imply that

$$\mu^\alpha = \chi^{\alpha\beta} h_\beta + \theta^{\alpha\beta} v_\beta, \quad h^\alpha = -4\pi n^\alpha n_\lambda \mu^\lambda, \quad \rho_1 = (\rho_0/\omega) k_\alpha v^\alpha,$$

$$\begin{aligned} &\{-i\omega\rho_0 \delta^{\alpha\beta} + k^2 n^\alpha n^\beta [i\rho_0 (a_0^2 + M_0 b)/\omega - M_0^2 (\nu_1 + \nu_2)]\} v_\beta \\ &+ \Big\{(1/2) M_0 \nu_1 k\omega \delta^{\alpha\beta} + n^\alpha n^\beta k M_0 \Big[(\nu_1/2 + \nu_2)\,\omega \\ &+ i \left(\frac{\rho_0 b}{M_0} + 4\pi + \beta_1 + \beta_2 \right) \Big] + (1/2) k\gamma \varepsilon^{\alpha\beta\lambda} n_\lambda \Big\} \mu_\beta = 0, \end{aligned} \tag{5.7.11}$$

where the components $\chi^{\alpha\beta}$ are determined by Eqs. (5.7.8) and (5.7.9), where we must put $\theta = \theta_{\|} = D = \tau^{-1} = \tau_{\|}^{-1} = 0$, while for $\theta^{\alpha\beta}$ we have

$$\begin{aligned} &\theta^{\alpha\beta} = \theta_1 \delta^{\alpha\beta} + \theta_2 n^\alpha n^\beta + i\theta_3 \varepsilon^{\alpha\beta\lambda} n^\lambda, \\ &\theta_1 = (1/2) k (-iM_0 \nu_1 \chi_1 + \gamma\chi_3), \\ &\theta_3 = (1/2) k (-iM_0 \nu_1 \chi_3 + \gamma\chi_1), \\ &\theta_2 = -\theta_1 - k (\chi_1 + \chi_2) \{iM_0 (\nu_1 + \nu_2) \\ &+ (1/\omega) [\rho_0 b - M_0/(\rho_0 I g^2)]\}. \end{aligned} \tag{5.7.12}$$

Equations (5.7.11) yield the dispersion equation for transverse waves:

$$k^2+\frac{4\rho_0}{\gamma^2+\nu_1^2M_0^2}\left[\beta_1+\frac{\omega}{g\,(\pm M_0+\rho_0 Ig\omega)}\right]=0, \tag{5.7.13}$$

or, solving for ω,

$$\omega=\mp M_0 g\,\frac{\rho_0\beta_1+(1/4)\,(\gamma^2+\nu_1^2M_0^2)\,k^2}{\rho_0+\rho_0 Ig^2\,[\rho_0\beta_1+(1/4)\,(\gamma^2+\nu_1M_0^2)\,k^2]}\,. \tag{5.7.14}$$

For transverse waves described by dispersion equation (5.7.14) we have (assuming that $n^\alpha=(0,\ 0,\ 1)$

$$\rho_1=0,\quad \mu^\alpha=(\mu^1,\ \mu^2,\ 0),\quad v^\alpha=(v^1,\ v^2,\ 0),\quad h^\alpha=0, \tag{5.7.15}$$

with μ and v related in the following manner:

$$\begin{aligned}&v^1\pm iv^2=0,\quad \mu^1\pm i\mu^2=0,\\ &v^1\mp iv^2=(ik/2\rho_0)\,(\nu_1M_0\pm i\gamma)\,(\mu^1\mp i\mu^2),\end{aligned} \tag{5.7.16}$$

with the signs in Eqs. (5.7.14) and (5.7.16) being adjusted.

If $\nu_1=\gamma=0$, the dispersion equation (5.7.14) gives the spin wave frequency

$$\omega_s=\mp M_0 g\beta_1/(1+\rho_0 Ig^2\beta_1), \tag{5.7.17}$$

where $v^1=v^2=v^3=0$. Hence, propagation of transverse acoustic waves in the liquid is due to mixed terms with M_α, ω_α, and $e_{\alpha\beta}$ in the energy U_m (for one, the gyromagnetic energy $\gamma M^\alpha\omega_\alpha$).

The dispersion curve for transverse waves is depicted schematically in Fig. 5.1 (the dashed line corresponds to $I=0$).

Equations (5.7.11) also give the dispersion equation for the longitudinal wave

$$\omega^2=a^2k^2/(1+e^2k^2), \tag{5.7.18}$$

where the coefficients a and e are determined by the following formulas:

$$\begin{aligned}&a^2=a_0^2+M_0b+\frac{(M_0-\rho_0^2g^2bI)\,[\rho_0b+M_0\,(4\pi+\beta_1+\beta_2)]}{\rho_0\,[1+\rho_0Ig^2\,(4\pi+\beta_1+\beta_2)]},\\ &e^2=\frac{Ig^2M_0^2\,(\nu_1+\nu_2)^2}{1+\rho_0Ig^2\,(4\pi+\beta_1+\beta_2)}\,.\end{aligned} \tag{5.7.19}$$

If the x^3-axis of the coordinate system is directed along the constant magnetization vector $\mathbf{M}_0$ so that $n^\alpha=(0,\ 0,\ 1)$, then for the longitudinal wave we have

$$v^\alpha=(0,\ 0,\ v),\quad \mu^\alpha=(0,\ 0,\ \mu),\quad h^\alpha=(0,\ 0,\ -4\pi\mu), \tag{5.7.20}$$

with

$$\mu=\frac{k}{\omega}\,v\,\frac{M_0-\rho_0Ig^2\,[\rho_0b+iM_0\,(\nu_1+\nu_2)\,\omega]}{1+\rho_0Ig^2\,(4\pi+\beta_1+\beta_2)}\,.$$

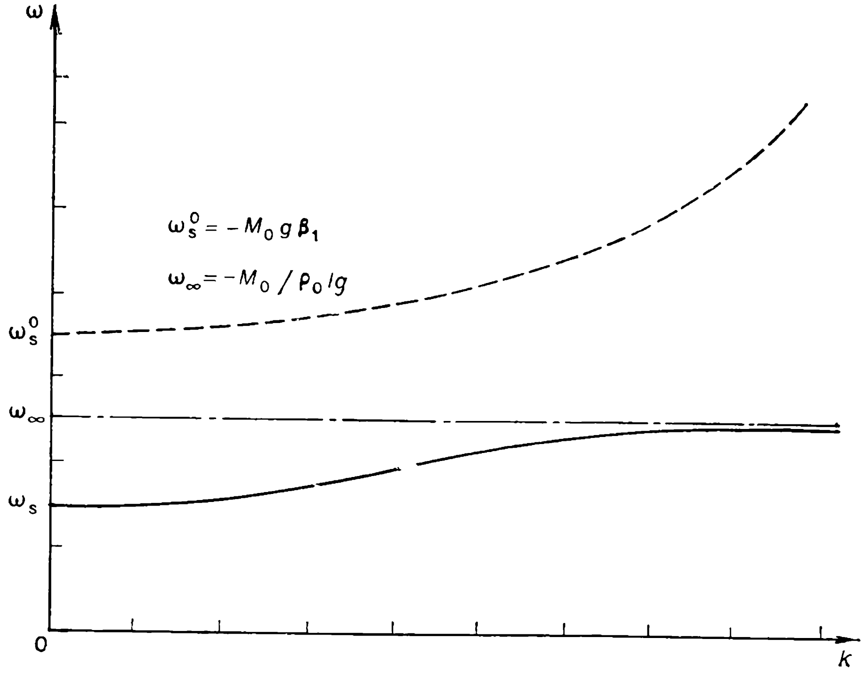

Fig. 5.1

The dispersion curve for longitudinal waves is depicted schematically in Fig. 5.2 (the dashed line corresponds to $I = 0$).

5.7.3. Finite Amplitude Plane Waves (Riemann Waves). We take an uncharged, nonconducting, and nonviscous magnetizable liquid with intrinsic moment of momentum $g^{-1}\mathbf{M}$ described by Eqs. (5.4.2) and (5.4.3) with $I = q^\alpha = i^\alpha = \rho_e = a^{\alpha\beta\lambda\theta} = \tau^{\alpha\beta\lambda\theta} = 0$, while the function U_m and the coefficients $\eta^{\alpha\beta}$ and $r^{\alpha\beta}$ are determined by relationships of the type

$$\rho U_m = \rho U_m(\rho,\ s,\ M),\quad r^{\alpha\beta} = (1/\tau g^2)(\delta^{\alpha\beta} - n^\alpha n^\beta),$$
$$\eta^{\alpha\beta} = \theta(\delta^{\alpha\beta} - n^\alpha n^\beta) + \theta_\parallel n^\alpha n^\beta.$$

Without derivation, we give a class of exact solutions for these equations in the form

$$v_3 = \pm \int (a_*/\rho)\, d\rho,\quad v^1 = v^2 = 0,\quad s = \text{const},$$
$$\begin{aligned} &M^1 = M^2 = 0,\quad M^3 = \rho m,\quad x^3 = t(v_3 \pm a_*) + F(\rho),\\ &H_1 = H_2 = 0,\quad H_3 = B_0 - 4\pi\rho m,\quad \Omega_1 = \Omega_2 = 0,\\ &E_1 = E_2 = E_3 = 0,\quad \Omega_3 = -g(B_0 - \partial\rho U_m/\partial M), \end{aligned} \qquad (5.7.21)$$

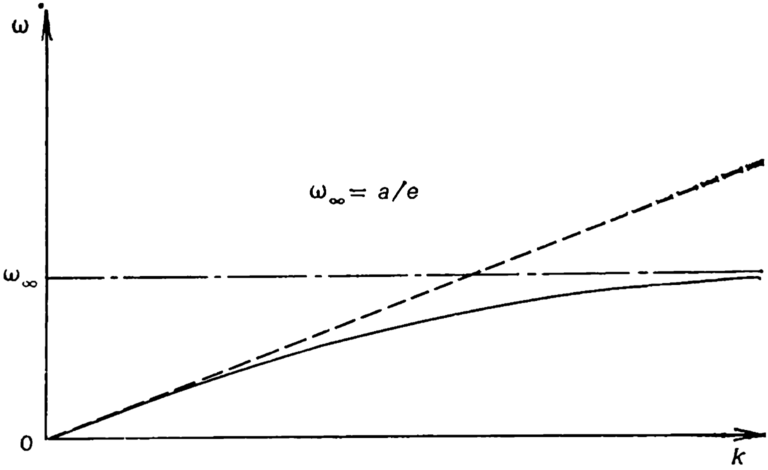

Fig. 5.2

where m and B_0 are arbitrary constants, and $F(\rho)$ an arbitrary differentiable function. The quantity a_* with the dimensions of velocity is determined by the relationship

$$a_*^2 = \rho\left(\frac{\partial^2 \rho U_m}{\partial \rho^2} + \frac{2}{\rho} M \frac{\partial^2 \rho U_m}{\partial \rho\, \partial M} + \frac{M^2}{\rho^2}\frac{\partial^2 \rho U_m}{\partial M^2}\right)$$

$$\equiv \left(\frac{\partial p}{\partial \rho}\right)_{M,\,B} + \frac{1}{\rho} M_\alpha \left(\frac{\partial p}{\partial M_\alpha}\right)_{\rho,\,B} + \frac{1}{2\rho} M_\alpha B^\alpha,$$

where for pressure p we have

$$p = \rho^2\, \partial U_m/\partial\rho + \rho M\, \partial U_m/\partial M - (1/2)\, M^\alpha B_\alpha.$$

Solution (5.7.21) determines plane waves of finite amplitude propagating along the x^3-axis, and a_* is the speed of the wave front.

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6

On Exact Solutions for Interacting Gravitational and Electromagnetic Fields

G. A. Alekseev, Cand.Sc. (Physics and Mathematics)
V. A. Steklov Institute of Mathematics, Academy
of Sciences of the USSR

INTRODUCTION

In Einstein's theory of gravitation, just as in any other nonlinear theory, exact solutions for field equations play an important role, since they allow a certain physical interpretation. The existence of a wide range of known exact solutions which describe relatively simple configurations of fields and the possibility (if it exists) of building nonlinear combinations of these solutions make it possible to construct various models of nonlinear interaction of fields from different sources with each other and with external fields.

This possibility is realized for the Einstein-Maxwell equations, which describe the interaction of a gravitational and an electromagnetic field, provided we take solutions that depend on two coordinates only.

Such solutions form two classes, both rich in physical contents. The first class includes solutions that depend on one time coordinate and one spatial, e.g. plane and cylindrical waves and solutions of a cosmological nature. The second class includes solutions that depend on two spatial coordinates. Among these of the greatest physical interest are those representing stationary fields with axial symmetry (which describe, among other things, external fields of black holes) as well as fields produced by various systems of sources and having complex multipole structures. For these classes of fields we can apply the ideas and methods of the inverse scattering theory [1]. This enables us to find in explicit form large families of solutions of field equations (by formal definition these solutions are called soliton solutions, or solitons).

Families of soliton solutions are built from a known exact solution depending only on two coordinates but in all other respects arbitrary. This solution serves as a background for the solitons. Aside from the freedom of choice of the background solution, the families of soliton solutions can have any (finite) number of free parameters. By using this freedom of choice of the background solution and parameters we can construct various solutions with preset physical contents.

The first application of methods of the inverse scattering theory to the Einstein equations in vacuum was in the works by Belinskii and Zakharov [2, 3], who found vacuum soliton solutions of the wave type and stationary with axial symmetry. In [4] the author of this paper found an explicit form of vacuum soliton solutions (in the form of determinants). The present paper discusses application of methods of the inverse scattering theory to integration of the Einstein-Maxwell equations in the electrovacuum case and examines the simplest soliton solutions.

6.1 THE EINSTEIN-MAXWELL EQUATIONS IN MATRIX FORM

We will write the metric and the four-potential of the electromagnetic field for the classes of electrovacuum fields that we consider here in the following form:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + g_{ab}\, dx^a\, dx^b, \quad A_i = \{0,\ 0,\ A_a\}, \tag{6.1.1}$$

where $\mu, \nu, \ldots = 0, 1$; $a, b, \ldots = 2, 3$; and the functions $g_{\mu\nu}$, g_{ab}, and A_a depend on the coordinates x^μ only and satisfy the electrovacuum Einstein-Maxwell equations.

By transforming the x^μ (while leaving alone the x^a) we can reduce the $g_{\mu\nu}$ to a conformally flat form $g_{\mu\nu} = -f\eta_{\mu\nu}$, with f positive and $\eta_{\mu\nu}$ constant. For stationary fields with axial symmetry we must put $\eta_{\mu\nu} = \mathrm{diag}\,(1,\ 1)$ if we employ cylindrical coordinates $x^\mu = \{\rho, z\}$ and $x^a = \{t, \varphi\}$. For fields that depend on one time coordinate and one spatial, i.e. $x^\mu = \{t, x\}$ and $x^a = \{y, z\}$, we can put, say, $\eta_{\mu\nu} = \mathrm{diag}\,(-1,\ 1)$.

The Einstein-Maxwell equations in a space free of sources can be written thus:

$$R_i{}^k = -2\left[F_{im}F^{km} - (1/4)\delta_i^k F_{lm}F^{lm}\right],\ \nabla_m F^{km} = 0,\quad F_{ik} = \partial_i A_k - \partial_k A_i \tag{6.1.2}$$

($\gamma = c = 1$; $R = 0$ for the electrovacuum case). For fields of type (6.1.1) these equations split into two groups: a closed system of equations for g_{ab} and A_a and equations for determining f (in terms of definite integrals) if g_{ab} and A_a are known. Hence, the problem is to find solutions to the closed system of equations for g_{ab} and A_a, which follow from (6.1.2) under condition (6.1.1). Here is a simple way of deriving such a system.

We start with the Maxwell equations $\nabla_k F^{ik} = 0$ and $\nabla_{[k} F_{mn]} = 0$ equivalent to $\nabla_{[k} F_{mn]} = 0$ and $\nabla_{[k} \overset{*}{F}_{mn]} = 0$, where the tensor $\overset{*}{F}_{ik}$ is dual to F_{mn}, i.e. $\overset{*}{F}_{mn} = (1/2)\, \varepsilon_{mnkl} F^{kl}$, with ε_{mnkl} the Levi-Civita symbol. The latter equations are equivalent to the equations $\nabla_{[k} \overset{+}{F}_{mn]} = 0$ for the self-dual Maxwell tensor $\overset{+}{F}_{mn} = F_{mn} + i\overset{*}{F}_{mn}$, which imply that $\overset{+}{F}_{mn}$ can be expressed in terms of the vector potential Φ_i that satisfies only the self-duality condition

$$\overset{+}{F}_{ik} = (i/2)\,\varepsilon_{ikmn}\overset{+}{F}{}^{mn}, \qquad \overset{+}{F}_{ik} = \partial_i\Phi_k - \partial_k\Phi_i. \tag{6.1.3}$$

These are the Maxwell equations in the form of duality equations that we will need in the future.

We can also write the Einstein equations in (6.1.2) in the form of duality equations. Since the metric (see (6.1.1)) has a two-dimensional Abelian symmetry group, i.e. admits of two commutative Killing vectors ξ^i_a, where $i = 0, 1, 2, 3$, and $a = 2, 3$ labels these vectors (in the metric (6.1.1) these vectors satisfy the following conditions: $\xi^\mu_a = 0$ and $\xi^b_a = \delta^b_a$), for each value of a we can introduce a bivector $K_a{}^{ij} = \nabla^i\xi^j_a - \nabla^j\xi^{ji}_a$ that by construction satisfies the equation $\nabla_{[i}K_{a\,jk]} = 0$, or (which is an equivalent form) $\nabla_j\overset{*}{K}_a{}^{ij} = 0$, where $\overset{*}{K}_a{}^{ij}$ is dual to $K_a{}^{ij}$; by employing the Killing equation we can find the divergence of $K_a{}^{ij}$, namely $\nabla_jK_a{}^{ij} = 2R^i{}_j\xi^j_a$ (in view of the Einstein equations in (6.1.2), the Ricci tensor $R^i{}_j$ can be expressed in terms of the electromagnetic field tensor). The equations for $K_a{}^{ij}$ that we have just obtained can be written in a unified complex form for the self-dual bivectors $\overset{+}{K}_a{}^{ij}$, namely $\nabla_{[i}\overset{+}{K}_{a\,jk]} = \nabla_{[i}\,(2\bar{\Phi}_l\xi^l_a\overset{+}{F}_{jk]})$. These equations imply that the self-dual bivectors

$$\overset{+}{H}_{aij} = \overset{+}{K}_{aij} - 2\bar{\Phi}_l\xi^l_a\overset{+}{F}_{ij}. \tag{6.1.4}$$

have potentials. These potentials, just as the Φ_k, must satisfy duality equations of the type (6.1.3):

$$\overset{+}{H}_{aij} = (i/2)\,\varepsilon_{ijmn}\overset{+}{H}_a{}^{mn}, \qquad \overset{+}{H}_{aij} = \partial_iH_{aj} - \partial_jH_{ai}. \tag{6.1.5}$$

The duality equations (6.1.3) and (6.1.5) together with expressions (6.1.4) in terms of (6.1.1) provide a closed system of equations

$$\begin{aligned}
\partial_\mu H_a{}^b &= -i\alpha^{-1}\varepsilon_\mu{}^\nu h_a{}^c\,\partial_\nu H_c{}^b,\\
\partial_\mu\Phi_a &= -i\alpha^{-1}\varepsilon_\mu{}^\nu h_a{}^c\,\partial_\nu\Phi_c,\\
\partial_\mu H_a{}^b &= \partial_\mu h_a{}^b - i\alpha^{-1}\varepsilon_\mu{}^\nu h_a{}^c\,\partial_\nu h_c{}^b - 2\bar{\Phi}^b\,\partial_\mu\Phi_a,
\end{aligned} \tag{6.1.6}$$

where $g_{ab} = h_a{}^c\epsilon_{cb}$, $\operatorname{Re}\Phi_a = A_a$, and $\varepsilon^{\mu\nu}$, ϵ_{ab}, and ϵ^{ab} are antisymmetric 2×2 matrices with unity in the upper right corners; the indices μ, ν, ... are lowered or raised by the matrix $\eta_{\mu\nu}$ and the inverse $\eta^{\mu\nu}$, while the indices a, b, ... are lowered or raised by the matrices ϵ_{ab} and ϵ^{ab}, respectively, e.g. $h_a{}^b = \epsilon^{bc}h_{ac}$ and $h_{ab} = h_a{}^c\epsilon_{cb}$. Equations (6.1.6) imply that $\det\,[h_{ab}] = \epsilon\alpha^2$ (where $\epsilon = \pm1$; the sign here is opposite to that of $\det\,[\eta_{\mu\nu}]$), with α satisfying the equation $\eta^{\mu\nu}\,\partial_\mu\partial_\nu\alpha = 0$, which is easily solved. In what follows we will take α as a fixed solution of the last equation such that

$\alpha = \text{const}$ is not an isotropic surface. In addition to α we will use a function β that to within a constant is determined by the equation $\partial_\mu \beta = \epsilon \varepsilon_\mu{}^\nu \, \partial_\nu \alpha$.

Equations (6.1.6) are the equations obtained by Kinnersley in [5], but here we have given a simpler derivation. It will be convenient to write (6.1.6) in an alternative complex matrix form. To this end we choose the quantities $H_{\mu a}{}^b \equiv \partial_\mu H_a{}^b$ and $\Phi_{\mu a} \equiv \partial_\mu \Phi_a$ as the new variables; in these variables the first two equations in (6.1.6) become

$$\begin{aligned} H_{\mu a}{}^b &= -i\alpha^{-1}\varepsilon_\mu{}^\nu h_a{}^c H_{\nu c}{}^b, \quad & \varepsilon^{\mu\nu}\partial_\mu H_{\nu a}{}^b &= 0, \\ \Phi_{\mu a} &= -i\alpha^{-1}\varepsilon_\mu{}^\nu h_a{}^c \Phi_{\nu c}, \quad & \varepsilon^{\mu\nu}\partial_\mu \Phi_{\nu a} &= 0. \end{aligned} \tag{6.1.7}$$

If we apply the operator $\varepsilon^{\gamma\mu}\partial_\gamma$, in view of the other pair of equations in (6.1.7) the left-hand sides vanish while the right-hand sides acquire terms with $\partial_\mu h_a{}^b$. These derivatives can be expressed in terms of $H_{\mu a}{}^b$ if we use a relationship that follows from the third equation in (6.1.6):

$$\begin{aligned} H_{\mu ab} - \bar{H}_{\mu ba} &= -2i\alpha^{-1}\varepsilon_\mu{}^\nu h_a{}^c \partial_\nu h_{cb} - 2i\beta_\mu \epsilon_{ab} \\ &- 2\bar{\Phi}_b \Phi_{\mu a} + 2\Phi_a \bar{\Phi}_{\mu b}. \end{aligned} \tag{6.1.8}$$

We must also allow for the relationships that follow directly from the duality equations (6.1.6):

$$\varepsilon^{\mu\nu}\bar{H}_\mu{}^c{}_a H_{\nu c}{}^b = 0, \quad \varepsilon^{\mu\nu}\bar{H}_\mu{}^c{}_a \Phi_{\nu c} = 0.$$

As a result, the equations we obtain are

$$\begin{aligned} &\eta^{\mu\nu}\partial_\mu H_{\nu a}{}^b + (i/2\alpha)\,\varepsilon^{\mu\nu}\,[H_{\mu a}{}^c + 2\bar{\Phi}^c \Phi_{\mu a}]\, H_{\nu c}{}^b = 0, \\ &\eta^{\mu\nu}\partial_\mu \Phi_{\nu a} + (i/2\alpha)\,\varepsilon^{\mu\nu}\,[H_{\mu a}{}^c + 2\bar{\Phi}^c \Phi_{\mu a}]\, \Phi_{\nu c} = 0, \\ &\varepsilon^{\mu\nu}\,\partial_\mu H_{\nu a}{}^b = 0, \quad \varepsilon^{\mu\nu}\,\partial_\mu \Phi_{\nu a} = 0. \end{aligned} \tag{6.1.9}$$

These equations can be written in a unified 3×3 matrix form:

$$\eta^{\mu\nu}\,\partial_\mu U_\nu + (i/2\alpha)\,\varepsilon^{\mu\nu} U_\mu U_\nu = 0, \quad \varepsilon^{\mu\nu}\,\partial_\mu U_\nu = 0, \tag{6.1.10}$$

where for each value of μ the quantities U_μ are 3×3 square matrices. It is easy to select expressions for the components of U_μ in terms of $H_{\mu a}{}^b$ and $\Phi_{\mu a}$ such that Eqs. (6.1.10) turn into corollaries of (6.1.6):

$$U_\mu = \begin{pmatrix} H_{\mu a}{}^b & \Phi_{\mu a} \\ 2\bar{\Phi}^c H_{\mu c}{}^b & 2\bar{\Phi}^c \Phi_{\mu c} \end{pmatrix}. \tag{6.1.11}$$

Using (6.1.11), we can express the Hermitian part of the third equation in (6.1.6) and the duality equations thus:

$$\Omega U_\mu - U_\mu^+ \Omega = -(1/2)\,\partial_\mu\,(G - 4i\beta\Omega), \tag{6.1.12}$$

$$GU_\mu = -4i\epsilon\alpha\varepsilon_\mu{}^\nu \Omega U, \tag{6.1.13}$$

(here the sign + as a superscript stands for Hermitian conjugation) where,

$$\Omega=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G=\begin{pmatrix} -4h^{ab}+4\Phi^a\overline{\Phi}^b & -2\Phi^a \\ -2\overline{\Phi}^b & 1 \end{pmatrix}. \tag{6.1.14}$$

Equations (6.1.10), (6.1.12), and (6.1.13) follow directly from (6.1.6). But do Eqs. (6.1.6) follow from them? Equations (6.1.12) and (6.1.13) can be thought of as restrictions on the choice of solutions of (6.1.10). Here (6.1.12) defines an additional Hermitian matrix G to within a constant term, while (6.1.13) defines the interrelationship of G and U_μ, which in the final analysis imposes restrictions on U_μ. The expression for G in (6.1.14) may be considered as the parametrization of a certain Hermitian matrix, but h^{ab} will then be an arbitrary 2×2 Hermitian matrix while it must be real; aside from this, the matrix element in the lower right corner of G is not necessarily unity. However, we will think of (6.1.12) as the definition not only of h^{ab} but of the function β as well, where β is defined in such a way that the imaginary part of h^{ab} (which is antisymmetric in view of the Hermiticity of h^{ab}) is zero. Such a definition of β coincides with that in terms of α given above. The matrix element in the lower right corner of G, which is constant in view of (6.1.12), can be taken to be unity by an appropriate choice of the constant of integration in (6.1.12). We must also require that the real part of Tr U_μ be zero.

Now it is easy to show that Eqs. (6.1.10), (6.1.12), and (6.1.13) together with the additional conditions

$$G^{**}=1, \quad \operatorname{Re}\operatorname{Tr} U_\mu = 0 \tag{6.1.15}$$

(in what follows the index * stands for the third row or column of 3×3 matrices, while the first two rows or columns are labeled by $a, b, \ldots$) are equivalent to the Einstein-Maxwell equations in the form (6.1.6). The obtained form of the field equations enables us to apply the ideas and methods of the inverse scattering theory to the construction of exact solutions.

6.2 BUILDING THE ASSOCIATED LINEAR SYSTEM AND THE REDUCTION CONDITIONS

The basic prerequisite for applying the methods of the inverse scattering theory to integrating the system of nonlinear equations is the building of a linear system that contains a free complex parameter λ. The linear system of equations must be overdetermined and its compatibility conditions (which are nonlinear equations for the coefficients of the linear system) must coincide with the system of nonlinear equations under consideration for all values of λ [1].

In what follows we will build such a linear system, i.e. a system for which Eqs. (6.1.10) are the compatibility conditions for all values of λ. Following Belinskii and Zakharov [2, 3] we will build a linear system of the general type for a 3×3 matrix function Ψ that depends on x^μ ($\mu = 0, 1$) and on the complex parameter λ:

$$D_\mu \Psi = \Lambda_\mu{}^\nu U_\nu \Psi, \tag{6.2.1}$$

where

$$D_\mu = \partial_\mu + P_\mu (\partial/\partial\lambda). \tag{6.2.2}$$

For fixed values of μ and ν the quantities P_μ and $\Lambda_\mu{}^\nu$ are functions (not matrix but numerical) of the variables x^μ and λ; for each value of μ the function U_μ is a 3×3 matrix depending on x^ν but not on λ. We wish to find such coefficients $P_\mu (x^\nu, \lambda)$ and $\Lambda_\mu{}^\nu (x^\nu, \lambda)$ so that the compatibility conditions (6.2.1) coincide with (6.1.10).

First we require that the operators (6.2.2) commute, i.e. $\varepsilon^{\mu\nu} D_\mu D_\nu = 0$. But (6.2.2) implies that $\varepsilon^{\mu\nu} D_\mu D_\nu = \varepsilon^{\mu\nu} D_\mu P_\nu (\partial/\partial\lambda)$; whence

$$\varepsilon^{\mu\nu} D_\mu P_\nu = 0. \tag{6.2.3}$$

Since the D_μ commute, the compatibility conditions (6.2.1) acquire the following form:

$$\varepsilon^{\mu\nu} D_\mu D_\nu \Psi = (\varepsilon^{\mu\nu} D_\mu \Lambda_\nu{}^\gamma U_\gamma + \Lambda_\nu{}^\gamma\, \varepsilon^{\mu\nu}\, \partial_\mu U_\nu$$
$$+ \varepsilon^{\mu\nu} \Lambda_\mu{}^\gamma \Lambda_\nu{}^\delta U_\gamma U_\delta)\, \Psi = 0.$$

This does not impose any restrictions on Ψ and, hence, for all values of λ we have

$$(\varepsilon^{\mu\nu} \Lambda_\nu{}^\gamma)\, \partial_\mu U_\gamma + (\varepsilon^{\mu\nu} D_\mu \Lambda_\nu{}^\gamma)\, U_\gamma - \varepsilon^{\gamma\delta} U_\gamma U_\delta \Lambda = 0, \tag{6.2.4}$$

where we employed the identity $\varepsilon^{\mu\nu} \Lambda_\mu{}^\gamma \Lambda_\nu{}^\delta = \Lambda \varepsilon^{\gamma\delta}$, with $\Lambda = \det [\Lambda_\mu{}^\nu]$. Now we want Eqs. (6.2.4) to be equivalent to (6.1.10). This can be achieved if we require that after we have divided Eqs. (6.2.4) by $2i\alpha\Lambda$ they become

$$[\eta^{\mu\nu}\, \partial_\mu U_\nu + (i/2\alpha)\, \varepsilon^{\mu\nu} U_\mu U_\nu] + F (x^\mu, \lambda)\, (\varepsilon^{\mu\nu}\, \partial_\mu U_\nu) = 0, \tag{6.2.5}$$

with $F (x^\mu, \lambda)$ an arbitrary function of x^μ and λ. If Eqs. (6.2.4) are satisfied for all values of λ, the expressions in square brackets nullify separately, which is equivalent to (6.1.10). Identifying coefficients of like terms in Eqs. (6.2.4) and (6.2.5) yields relationships that together with (6.2.3) uniquely determine P_μ and $\Lambda_\mu{}^\nu$:

$$\varepsilon^{\mu\nu} \Lambda_\nu{}^\gamma = 2i\alpha\Lambda\, (\eta^{\mu\gamma} + F\varepsilon^{\mu\gamma}), \tag{6.2.6}$$

$$\varepsilon^{\mu\nu} D_\mu \Lambda_\nu{}^\gamma = 0. \tag{6.2.7}$$

Equation (6.2.6) implies that $\Lambda_\mu{}^\nu = 2i\alpha\Lambda\, (F\delta^\nu{}_\mu + \epsilon\varepsilon_\mu{}^\nu)$, whence $\Lambda^{-1} = 4\epsilon\alpha^2\, (1 - \epsilon F^2)$. Equation (6.2.7) is used to determine the

P_μ, with (6.2.3) satisfied automatically. Hence, for $\Lambda_\mu{}^\nu$ and P_μ we have the following expressions:

$$\Lambda_\mu{}^\nu = \frac{i}{2\alpha}\,\frac{\epsilon F\delta_\mu^\nu + \varepsilon_\mu^\nu}{1-\epsilon F^2}\,, \quad P_\mu = -\frac{\partial_\mu(\alpha F+\beta)}{\alpha\,(\partial F/\partial\lambda)}\,.$$

It is expedient to choose $F\,(x^\mu, \lambda)$ such that P_μ vanishes. To this end we put $w = \alpha F + \beta = \text{const}$ and take w as the new complex parameter. Then instead of (6.2.1) we have

$$\partial_\mu\Psi = \Lambda_\mu{}^\nu U_\nu\Psi, \quad \Lambda_\mu{}^\nu = \frac{1}{2i}\,\frac{(w-\beta)\,\delta_\mu^\nu + \epsilon\alpha\varepsilon_\mu{}^\nu}{(w-\beta)^2-\epsilon\alpha^2}\,. \tag{6.2.8}$$

This system has nondegenerate solutions if and only if the matrices U_ν satisfy Eqs. (6.1.10).

Let us now determine the reduction conditions, i.e. the additional restrictions imposed on the solutions of the linear system for Ψ that are introduced by (6.1.12), (6.1.13), and (6.1.15).

If we multiply Eq. (6.1.12) by $4i\,(w-\beta)$ and add the result to the anti-Hermitian part of (6.1.13), the resulting equation can be written as

$$\partial_\mu W + \Lambda_\mu{}^\nu\,(WU_\nu - U_\nu^+ W) = 0, \quad W = G + 4i\,(w-\beta)\,\Omega. \tag{6.2.9}$$

Now if we use (6.2.8) to express $\Lambda_\mu{}^\nu U_\nu$ in terms of Ψ as $\Lambda_\mu{}^\nu U_\nu = D_\mu\Psi\cdot\Psi^{-1}$ and substitute the result into (6.2.9), we find that $\partial_\mu\,(\Psi^+ W\Psi) = 0$, which is equivalent to

$$\Psi^+ W\Psi = K\,(w), \quad K^+\,(w) = K\,(w), \tag{6.2.10}$$

where Ψ^+ is by definition $\overline{\Psi^T}\,(\overline{w})$, i.e. a Hermitian-conjugate matrix at a point in the complex w-plane that is complex conjugate to w, and $K\,(w)$ is an arbitrary nondegenerate Hermitian matrix. Note that Ψ is determined via the system (6.2.8) to within multiplication from the right by an arbitrary matrix depending on w. This enables us in principle to reduce $K\,(w)$ to a canonical form independent of w. The idea behind the restriction (6.2.10) is that $W = (\Psi^+)^{-1}K\Psi^{-1}$ is a Hermitian matrix linear in w, and the coefficient of $w-\beta$ is $4i\Omega$. The absolute term then gives the matrix G and, hence, is under the same restrictions: the quantity h^{ab}, which determines β, must be real and $G^{**} = 1$, which is equivalent to $W^{**} = 1$.

Therefore, to Eq. (6.2.8) and condition (6.2.10) we must add the conditions that W be linear in w, $W^{**} = 1$, $\operatorname{Re}\operatorname{Tr} U_\mu = 0$, and the Hermitian part of (6.1.13).

Following the worked-out procedure of applying the method of inverse scattering, we change variables in the built linear system. Instead of Ψ we introduce a matrix χ by the following relationship: $\Psi = \chi\overset{\circ}{\Psi}$, where $\overset{\circ}{\Psi}$ is a 3×3 nonsingular matrix corresponding to an arbitrarily chosen but known exact solution of Eqs. (6.1.6). The matrix $\overset{\circ}{\Psi}$ is found from Eqs. (6.2.8), where for U_μ we substitute $\overset{\circ}{U}_\mu$ calculated via (6.1.11) with the chosen exact solution. This exact solution will play the role of background for building the solitons, and the sought solution will describe the interaction of the solitons with the background.

Let us now formulate all the conditions imposed on Ψ as the conditions imposed on the matrices χ. According to (6.2.8), χ must satisfy the equation

$$\partial_\mu \chi = \Lambda_\mu{}^\nu \ (U_\nu \chi - \chi \overset{\circ}{U}_\nu). \tag{6.2.11}$$

Assuming that for all solutions the canonical form of matrix $K\ (w)$ is the same, we find from (6.2.10) that

$$\chi^+ W \chi = \overset{\circ}{W}. \tag{6.2.12}$$

Next, we assume that χ is analytic in w as $w \to \infty$. From (6.2.11) and (6.2.8) it follows that the limit of χ as $w \to \infty$ is independent of x^μ; we can assume, without loss of generality, that this limit is equal to a unit matrix. Then as $w \to \infty$,

$$\chi = I + (1/w)\, R + \ldots . \tag{6.2.13}$$

The form of this expansion together with (6.2.12) and the condition that matrix W be regular in any finite part of the complex w-plane suggest that both W and $\overset{\circ}{W}$ are linear in w with the coefficient $4i\Omega$ and that $W^{**} = 1$. Instead of requiring that $\operatorname{Re}\operatorname{Tr} U_\mu = 0$ we can require that $\operatorname{Tr} U_\mu = \operatorname{Tr} \overset{\circ}{U}_\mu$, which in view of (6.2.11) means that $\det \chi$ is independent of x^μ. In addition, (6.2.12) yields $\det (\chi\chi^+) = 1$. These conditions imply that in view of (6.2.13) $\operatorname{Tr} R$ is independent of x^μ and its real part is zero.

Hence, the problem has been reduced to building solutions for (6.2.11) that satisfy conditions (6.2.12), (6.2.13), $\partial_\mu \det \chi = 0$, $\det (\chi\chi^+) = 1$, and the Hermitian part of Eqs. (6.1.13). The sought solution can then be determined by the following formulas:

$$U_\mu = \overset{\circ}{U}_\mu + 2i\partial_\mu R, \tag{6.2.14}$$

$$G = \overset{\circ}{G} - 4i\,(\Omega R + R^+\Omega) + 4i\beta_0\Omega, \quad \beta = \overset{\circ}{\beta} + \beta_0, \tag{6.2.15}$$

where β_0 is a constant determined from the condition that h^{ab} is real, which is a consequence of (6.2.15), and $\overset{\circ}{\beta}$ is the function β for the background solution $\overset{\circ}{U}_\mu$, $\overset{\circ}{G}$.

If we employ (6.1.14), we have the solution in explicit form:

$$
\begin{aligned}
&\Phi_a = \mathring{\Phi}_a + 2iR_a{}^* \\
&h_{22} = \mathring{h}_{22} + i\,(\overline{R}_2{}^3 - R_2{}^3) + \Phi_2\overline{\Phi}_2 - \mathring{\Phi}_2\overline{\mathring{\Phi}}_2, \\
&h_{23} = \mathring{h}_{23} + (i/2)\,(R_2{}^2 - \overline{R}_2{}^2 - R_3{}^3 + \overline{R}_3{}^3) + (1/2)\,(\Phi_2\overline{\Phi}_3 + \overline{\Phi}_2\Phi_3) \\
&- (1/2)\,(\mathring{\Phi}_2\overline{\mathring{\Phi}}_3 + \overline{\mathring{\Phi}}_2\mathring{\Phi}_3), \\
&h_{33} = \mathring{h}_{33} + i\,(R_3{}^2 - \overline{R}_3{}^2) + \Phi_3\overline{\Phi}_3 - \mathring{\Phi}_3\overline{\mathring{\Phi}}_3,
\end{aligned}
\qquad (6.2.16)
$$

where according to (6.1.1) a is either 2 or 3, and * stands for the third column in matrix R.

6.3 SOLITON SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS

In the general case, the problem of building the matrix functions χ, U_μ, and W that satisfy the above-mentioned conditions can be reduced, by a technique worked out in [1, 2], to the Riemann problem of analytic continuation of a complex-valued matrix function to the interior and exterior of a contour with given matching conditions on the contour; this is equivalent to solving a matrix linear integral equation. The solution for this problem (for fixed normalization of the matrix functions) is unique if we require that the sought solution be nondegenerate (i.e. the determinant of the matrix function does not vanish) everywhere in the exterior of the contour. But in general the nondegeneracy may be violated at some points of the exterior. An important case here is when the matrix function has a finite number of points in the exterior of the contour where its determinant vanishes and the inverse matrix function has simple poles, and the matching conditions on the contour are simply the continuity condition (i.e. the function that defines the matching is unity). In this case the complete solution can be obtained by purely algebraic methods and is called a soliton solution. In what follows we will build soliton solutions mainly by using the technique developed in [2]. Our method, which differs very little from the above-mentioned, enables building soliton solutions of Eqs. (6.2.11)-(6.2.13) under the additional restrictions on χ listed above. A direct verification shows that conditions (6.1.13) are automatically satisfied by the constructed solutions.

In agreement with what we have just said, let us assume that the matrices χ and χ^{-1} have the following form:

$$\chi = I + \sum_{k=1}^{N} R_k/(w - w_k), \qquad \chi^{-1} = I + \sum_{k=1}^{N} S_k/(w - \widetilde{w}_k), \qquad (6.3.1)$$

where N is the number of solitons, R_k and S_k are unknown residues depending on x^μ, and the poles w_k and $\tilde{w}_k$ are generally different and also depend on x^μ.

The matrices R_k and S_k, obviously, are not independent of each other. The conditions $\chi\chi^{-1} = \chi^{-1}\chi = I$ imply that

$$R_k\chi^{-1}(w_k) = \chi^{-1}(w_k)R_k = S_k\chi(\tilde{w}_k) = \chi(\tilde{w}_k)S_k = 0$$

for each $k=1, 2, \ldots, N$. This shows that R_k and S_k are degenerate. Without loss of generality we can assume that both R_k and S_k are of rank 1 and, hence, can be represented by a tensor product of pairs of vectors:

$$R_k = n_k \otimes m_k, \quad S_k = p_k \otimes q_k. \tag{6.3.2}$$

These vectors satisfy the following relationships:

$$\begin{aligned} m_k + \sum_{l=1}^{N} \frac{(m_k\cdot p_l)}{w_k - \tilde{w}_l} q_l = 0, \quad q_k + \sum_{l=1}^{N} \frac{(q_k\cdot n_l)}{\tilde{w}_k - w_l} m_l = 0, \\ n_k + \sum_{l=1}^{N} \frac{(n_k\cdot q_l)}{w_k - \tilde{w}_l} p_l = 0, \quad p_k + \sum_{l=1}^{N} \frac{(p_k\cdot m_l)}{\tilde{w}_k - w_l} n_l = 0. \end{aligned} \tag{6.3.3}$$

Here $(\cdot)$ stands for the scalar product of two vectors. For the subsequent discussion it is expedient to introduce an $N \times N$ matrix Γ with elements

$$\Gamma_{kl} = (p_k\cdot m_l)/(w_l - \tilde{w}_k). \tag{6.3.4}$$

We can now express n_k and q_k via (6.3.4) in terms of vectors p_k, m_k and matrix Γ:

$$n_k = \sum_{l=1}^{N} (\Gamma^{-1})_{kl} p_l, \quad q_k = -\sum_{l=1}^{N} (\Gamma^{-1})_{lk} m_l. \tag{6.3.5}$$

Hence, the only arbitrariness in (6.3.1) is in the choice of w_k and $\tilde{w}_k$ and the vectors p_k and m_k. We now turn to the conditions (6.2.11) and (6.2.12) which matrices (6.3.1) must satisfy. Substituting (6.3.1) into (6.2.11) we find that

$$\partial_\mu w_k = \partial_\mu \tilde{w}_k = 0, \tag{6.3.6}$$

i.e. w_k and $\tilde{w}_k$ are arbitrary constants. Condition (6.3.6) means that there are no second-order poles in the expressions we obtain. Then

$$\Lambda_\mu{}^\nu U_\nu = \partial_\mu\chi\cdot\chi^{-1} + \Lambda_\mu{}^\nu \chi \overset{\circ}{U}_\nu \chi^{-1} \tag{6.3.7}$$

has first-order poles and we obtain

$$\partial_\mu m_k + \Lambda_\mu{}^\nu (w_k)\, m_k\cdot \overset{\circ}{U}_\nu = 0,$$
$$\partial_\mu p_k - \Lambda_\mu{}^\nu (\tilde{w}_k)\, \overset{\circ}{U}_\nu\cdot p_k = 0, \tag{6.3.8}$$

where the dot stands for contraction with respect to corresponding indices of the matrices and vectors. Generally speaking, in Eqs. (6.3.8) the right-hand sides must contain terms proportional to m_k and p_k, respectively, but the absence of these terms does not lead to any loss in solutions. The solutions of Eqs. (6.3.8) are expressed in terms of the matrix $\overset{\circ}{\Psi}$ determined from Eq. (6.2.8) with $\overset{\circ}{U}_\mu$ substituted for U_μ:

$$m_l = k_l\cdot \overset{\circ}{\Psi}^{-1} (w_l), \quad p_k = \tilde{\Psi} (\tilde{w}_k)\cdot l_k, \tag{6.3.9}$$

where k_l and l_k are arbitrary constant complex three-dimensional vectors (the indices $k, l, \ldots = 1, 2, \ldots, N$ label these vectors). Relationships (6.3.1), (6.3.2), (6.3.4), (6.3.5) and (6.3.9) define very broad classes of exact solutions of Eqs. (6.1.10) with a large number of arbitrary constants; these constants are the sets w_k and $\tilde{w}_k$ and two sets of vectors k_l and l_k.

All the additional conditions that χ must satisfy lead to relationships between these sets of constants. For instance, if we write (6.2.12) as $\chi^+ W = \overset{\circ}{W}\chi^{-1}$, we find that the poles $\tilde{w}_k$ are the complex conjugates of the w_k, i.e. $\tilde{w}_k = \overline{w}_k$. Taking (6.2.12) in the form $W = (\chi^+)^{-1}\overset{\circ}{W}\chi^{-1}$ and demanding that the right-hand side in this relationship have no singularities (poles), we come to equations for the vectors n_k:

$$\sum_{l=1}^{N} (m_l\cdot \overset{\circ}{W}^{-1} (\overline{w}_k)\cdot \overline{m}_k)\, n_l/(w_l - \overline{w}_k) = \overset{\circ}{W}^{-1} (\overline{w}_k)\cdot \overline{m}_k. \tag{6.3.10}$$

These equations are equivalent to Eqs. (6.3.4), (6.3.5), and (6.3.9) provided that the constant vectors k_l and l_k are related in the following manner:

$$l_k = \overset{\circ}{K}^{-1} (\overline{w}_k)\, \overline{k}_k. \tag{6.3.11}$$

In all other respects the choice of the constants is arbitrary. The matrix R is expressed thus:

$$R = \sum_{k=1}^{N} n_k \otimes m_k, \tag{6.3.12}$$

where the vectors n_k and m_k are given by the formulas

$$m_k = k_h \cdot \overset{\circ}{\Psi}{}^{-1}(w_k), \qquad n_k = \sum_{l=1}^{N} (\Gamma^{-1})_{kl} p_l,$$
$$\Gamma_{kl} = (m_l \cdot p_k)/(w_l - \bar{w}_k), \quad p_k = \overset{\circ}{W}{}^{-1}(\bar{w}_k) \cdot \bar{m}_k. \tag{6.3.13}$$

These formulas fully determine the soliton solutions for the Einstein-Maxwell equations if we have the background field, the sets of arbitrary complex constants w_k, and the sets of constant complex three-dimensional vectors k_l.

Next we write the matrix R for soliton solutions in explicit form. To this end we note that each element of R (in what follows we number the rows and columns of this matrix by labels A, B, . . .; as usual, the first two rows and columns we label a, b, . . ., while the third row and third column are labeled *) can be represented by the sum of elements of a matrix that is the inverse of an $N \times N$ matrix whose elements can be calculated from the background solution and the sets of arbitrary constants w_k and k_l. This representation is similar to that used in [4] for finding an explicit form (in terms of determinants) for vacuum soliton solutions:

$$R_A{}^B = a_A b^B \sum_{k,l=1}^{N} (\Delta^{-1})^{kl}, \quad \Delta_{kl} = \Gamma_{kl} a_A b^B / m_l{}^B p_{kA}. \tag{6.3.14}$$

Here we introduced two three-dimensional vectors a_A and b^B with nonzero components. Since R does not generally depend on the choice of a_A and b^B, the components may be chosen arbitrarily (e.g. be functions of the coordinates), but a successful choice may simplify calculations considerably. Using the formula for the sum of elements of a matrix inverse to a given matrix, namely

$$\sum_{k,l=1}^{N} (\Delta^{-1})_{kl} = \det \|\Delta_{kl} + 1\| / \det \|\Delta_{kl}\| - 1,$$

from (6.3.14) we find that

$$R_A{}^B = a_A b^B \left(\frac{\det \|\Gamma_{kl} + (a_A b^B)^{-1} p_{kA} m_l{}^B\|}{\det \|\Gamma_{kl}\|} - 1 \right). \tag{6.3.15}$$

Here we do not sum over repeated indices A, B, ...

In the above N-soliton solution determined by R via (6.3.15) there are N arbitrary complex parameters w_k and $3N$ complex parameters, the components of the three-dimensional vectors k_l (however, not all components are significant). We can easily see from the general formula (6.3.15) that each of the vectors k_l may be multiplied by an arbitrary constant with no effect on $R_A{}^B$. Hence, we can assume, for instance, that $k_l = \{1, c_l, d_l\}$, where c_l and d_l constitute a set of $2N$ complex parameters that are significant. We then find that the electrovacuum N-soliton solution contains $3N$ arbitrary significant complex parameters.

This ends the description of the algorithm for building soliton solutions of the Einstein-Maxwell equations. Below we will study some simple soliton solutions obtained by applying the above algorithm.

6.4 ONE-SOLITON SOLUTIONS WITH MINKOWSKI'S SPACE-TIME AS BACKGROUND

Let us consider the simplest case, where the background for the solitons is Minkowski's space-time. In this case f and g_{ab} can be chosen thus:

$$\mathring{f}=1, \quad \mathring{g}_{ab}=\begin{pmatrix} -\epsilon & 0 \\ 0 & -\alpha^2 \end{pmatrix}, \quad \mathring{\Phi}_a=0. \tag{6.4.1}$$

This notation encompasses several interesting cases. At $\alpha=\rho$ we have a flat metric in cylindrical coordinates, while at $\alpha=t$ we arrive at coordinates that cover a part of Minkowski's space-time and expand with time, just as in the Kasner cosmological model (however, in contrast to Kasner's model the singularity at $t=0$ is purely coordinate).

For $\epsilon=1$ the soliton solutions that we wish to build will depend on time and one spatial coordinate. If in this case $\alpha=\rho$, we have cylindrical waves, while if $\alpha=t$, we have, for instance, solutions of the cosmological type with perturbations in the form of plane waves. For $\epsilon=-1$ and $\alpha=\rho$, we have stationary axisymmetric soliton solutions.

Calculations involving the background solution (6.4.1) yield

$$\mathring{U}_\mu=\begin{pmatrix} 2i\beta_\mu & 0 & 0 \\ -2\alpha\alpha_\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathring{W}=\begin{pmatrix} 4\alpha^2 & 4i(w-\beta) & 0 \\ -4i(w-\beta) & 4\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathring{\Psi}=\begin{pmatrix} \Sigma^{-1} & 0 & 0 \\ i\epsilon(w-\beta)\Sigma^{-1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathring{K}=4\begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1/4 \end{pmatrix},$$

where $\Sigma(\alpha, \beta, w)$ is the continuous branch of a two-valued function determined by the equality $\Sigma^2=(w-\beta)^2-\epsilon\alpha^2$.

For a one-soliton solution ($N=1$) we put $k^A=\{1, c, d\}$. We then have the following expressions for the vectors m^A and n_A:

$$m^A=\begin{pmatrix} \Sigma-i\epsilon(w-\beta)c \\ c \\ d \end{pmatrix}, \quad n_A=\frac{w-\bar{w}}{D}\begin{pmatrix} 1 \\ i\epsilon(\bar{w}-\beta)-\bar{c}\Sigma(\bar{w}) \\ -4\epsilon\bar{d}\Sigma(\bar{w}) \end{pmatrix},$$

with

$$D = \Sigma\,(w) - (c\bar{c} + 4\epsilon\, d\bar{d})\,\Sigma\,(\bar{w}) + 2\,\epsilon\sigma c, \quad w = i\sigma + \beta_1.$$

In these formulas w determines the position of a pole (since $N = 1$, the label 1 on k^A and w is dropped). For the sake of definiteness we chose the branch of $\Sigma\,(w)$ on which $\overline{\Sigma\,(w)} = \Sigma\,(\bar{w})$.

For this solution we have

$$R = \frac{2i\sigma}{D}\begin{pmatrix} F_1 & c & d \\ F_1F_2 & cF_2 & dF_2 \\ -4\epsilon\,\bar{d}\,\bar{\Sigma}F_1 & -4\epsilon\bar{d}c\,\bar{\Sigma} & -4\epsilon d\bar{d}\,\bar{\Sigma} \end{pmatrix},$$

with

$$F_1 = \Sigma - i\epsilon\,(w - \beta)\,c, \quad F_2 = i\epsilon\,(\bar{w} - \beta) - \bar{c}\bar{\Sigma}.$$

According to the general formulas (6.2.16), all components of the metric h_{ab} and electromagnetic field Φ_a may be determined via the formula for R, but here we will confine ourselves to calculating h_{22} and Φ_2, the more so that these components contain all information concerning the solution. These two components are

$$\Phi_2 = -4\sigma\, d/D, \quad h_{22} = [-\epsilon D\bar{D} + 2\sigma\,(c\bar{D} + \bar{c}D) + 16d\bar{d}\sigma^2]/D\bar{D}. \tag{6.4.2}$$

Let us now consider the cases for different ϵ separately. At $\epsilon = -1$ the one-soliton solution is stationary and axisymmetric. We must put $\alpha = \rho$, $\beta - \beta_1 = z$, $x^2 = t$, and $x^3 = \varphi$. We introduce new parameters m, a, b, and c in the following manner:

$$\begin{gathered} m - ib = -\frac{\sigma c}{1 - c\bar{c} + 4d\bar{d}}, \quad a = -\frac{\sigma\,(1 + c\bar{c} - 4d\bar{d})}{1 - c\bar{c} + 4d\bar{d}}, \\ e = -\frac{2\sigma d}{1 - c\bar{c} + 4d\bar{d}}, \quad -\sigma^2 = m^2 + b^2 - a^2 - e\bar{e}, \end{gathered} \tag{6.4.3}$$

and change the coordinates from ρ and z to r and θ:

$$\rho = \sqrt{(r - m)^2 + \sigma^2}\sin\,\theta, \quad z = (r - m)\cos\theta.$$

As a result we have

$$\begin{gathered} \Sigma = r - m - i\sigma\cos\theta, \quad D = \frac{2\sigma}{\sigma - a}\,(r + ia\cos\theta - ib), \\ \Phi_t = \frac{e}{r + ia\cos\theta - ib}, \quad h_{tt} = \frac{r^2 - 2mr - b^2 + e\bar{e} + a^2\cos^2\theta}{r^2 + (a\cos\theta - b)^2}. \end{gathered} \tag{6.4.4}$$

These formulas coincide exactly with those for the known exact solution for the field of a charged rotating black hole (the Kerr-Newman solution) of mass m, angular momentum a, the NUT parameter b, electric charge Re e, and magnetic charge Im e. However, the expression for σ^2 in (6.4.3) shows that this family of solutions constitutes the superextremal part of the Kerr-Newman family of solutions, i.e. the part for which there is no horizon. A transition to solutions with a horizon can be achieved by analytic continuation of the solution with respect to the parameters to the region $m^2 + b^2 - a^2 - e\bar{e} \geqslant 0$.

We now go on to the case with $\epsilon = 1$. We put $\alpha = t$, $\beta - \beta_1 = z$, $x^2 = x$, and $x^3 = y$. Then in (6.4.2)

$$\begin{aligned} D &= \operatorname{sgn} z\,[H - i\sigma z/H - (c\bar{c} + 4d\bar{d})(H + i\sigma z/H)] + 2\sigma c, \\ H &= \{(z^2 - \sigma^2 - t^2)/2 + [(z^2 - \sigma^2 - t^2)^2 + 4\sigma^2 t^2]^{1/2}/2\}^{1/2}. \end{aligned} \qquad (6.4.5)$$

In these coordinates the explicit expression for Σ is

$$\Sigma = \operatorname{sgn} z\,(H - i\sigma z/H). \qquad (6.4.6)$$

How does this solution behave with time? At $t = 0$ there is a coordinate singularity in the background solution. For this reason we restrict our discussion to the case of $t > 0$, since the background metric as well as solution (6.4.2) are completely regular.

For small positive values of t we have

$$H = |z|\left[1 - \frac{t^2}{2(z^2 + \sigma^2)} + \ldots\right],$$

$$\Sigma = z - i\sigma - \frac{t^2(z + i\sigma)}{z^2 + \sigma^2} + \ldots,$$

$$\begin{aligned} D = {} & z(1 - c\bar{c} - 4d\bar{d}) - i\sigma(1 + c\bar{c} + 4d\bar{d}) \\ & - \frac{t^2}{2(z^2 + \sigma^2)}[z(1 - c\bar{c} - 4d\bar{d}) + i\sigma(1 + c\bar{c} + 4d\bar{d})] + \ldots . \end{aligned}$$

Substituting into (6.4.2), we find that

$$\begin{aligned} \Phi_x &= \frac{e}{z + il} + O(t^2), \\ h_{xx} &= -1 + \frac{k\sigma(z\cos\delta + l\sin\delta) + e\bar{e}}{z^2 + l^2} + O(t^2), \end{aligned} \qquad (6.4.7)$$

with the constants k, l, e, and δ defined thus:

$$ke^{i\delta} = \frac{4c}{1 - c\bar{c} - 4d\bar{d}}, \quad l = -\frac{\sigma(1 + c\bar{c} + 4d\bar{d})}{1 - c\bar{c} - 4d\bar{d}}, \quad e = -\frac{4\sigma d}{1 - c\bar{c} - 4d\bar{d}}.$$

As we can easily see from (6.4.7), the components Φ_x and h_{xx} in the built solution describe at an initial moment close to $t = 0$ a perturbation localized along the z axis. Indeed, as $z \to \pm\infty$ both tend to the corresponding values for the background solution, i.e. the perturbation vanishes: $h_{xx} \to -1$ and $\Phi_x \to 0$. All other components of the metric and the electromagnetic field behave in a similar manner. Let us now study the behavior of the solution with time.

Equations (6.4.5) and (6.4.6) imply that for each fixed z and as $t \to \infty$ all deviations from the background solution vanish. To find the asymptotic behavior of the perturbation we are considering as $t \to \infty$, we go over to new coordinates: $u = t - z$ and $v = t + z$ (retarded and advanced times), and consider the behavior of the solution for (1) $v \to \infty$ and u finite and (2) $u \to \infty$ and v finite.

In the first case, i.e. as $v \to \infty$, we have

$$\begin{aligned} \Phi_x &= \frac{2e\sqrt{B(u)}}{v^{1/2}\,[2B(u)+il]} + O(v^{-3/2}), \\ h_{xx} &= -1 + \frac{2k\sigma}{\sqrt{v}}\,\frac{\sqrt{B(u)}\,[2B(u)\cos\delta + l\sin\delta]}{4B^2(u)+l^2} + O(v^{-3/2}), \end{aligned} \tag{6.4.8}$$

where $B(u) = \sqrt{u^2 + \sigma^2} - u$. This implies that as $v \to \infty$, the shape of the perturbation depends only on the retarded time u while the amplitude of the perturbation decreases as $v^{-1/2}$. This means that for large v's such perturbations may be taken as waves that propagate along the light cone $u =$ const. From (6.4.8) we can also see that $\Phi_x \to 0$ and $h_{xx} \to -1$ as $u \to \pm\infty$, i.e. for large v's the solution is localized with respect to the retarded time u. For the case $u \to \infty$ we arrive at a localized wave that is similar to (6.4.8) but propagates along isotropic surfaces $v =$ const.

Hence, the one-soliton solution (6.4.5) with the Minkowski space-time as background describes the decay of a localized perturbation into two waves propagating in opposite directions. For large t's these waves propagate with the speed of light. The decrease in their amplitudes is due to the increase in their front areas, which are given by $(\det \| h_{ab} \|)^{1/2} = t \sim v$ as $v \to \infty$.

We note that by varying the parameters in the soliton solutions we can change the shape of the initial perturbation. If $N > 1$, then by selecting certain values of the parameters we can not only make the shape of the initial localized perturbation more complicated but produce at time zero several isolated perturbations along the z axis that in the course of time decay into solitary waves propagating in opposite directions. The characteristic feature of solitons manifests itself in the propagation and interaction of the "decay products" of the initial perturbations; namely, the product waves passing through each other and traveling to infinity have the same shape as if there were no counterpropagating waves.

We can interpret the solution (6.4.2) differently if we put $\alpha = \rho$, $\beta - \beta_1 = t$, $x^2 = z$, and $x^3 = \varphi$. Then it describes a cylindrical wave propagating from infinity, approaching the axis, reflecting from the axis, and going to infinity. Soliton cylindrical waves exhibit the same properties in their propagation and interaction with each other as do the plane waves described above.

6.5 INTERACTION OF SOLITONS WITH A UNIFORM ELECTROMAGNETIC FIELD

The algorithm for building electrovacuum soliton solutions proposed in this work enables us to describe the interaction of various soliton configurations with external electrovacuum fields. For this we must take the solution that describes the external field as the background.

As an example we consider the interaction of a one-soliton perturbation with a uniform electromagnetic field. A uniform electromagnetic field is described by the following exact solution of the Einstein-Maxwell equation:

$$\overset{\circ}{h}_{ab}=\begin{pmatrix} -\epsilon W^2 & 0 \\ 0 & -\alpha^2/W^2 \end{pmatrix}, \overset{\circ}{\Phi}_a=B_0\begin{pmatrix} 2\beta \\ i\alpha^2/W \end{pmatrix}, \tag{6.5.1}$$

where $W = 1 + k\alpha^2$, $k = B_0\overline{B}_0$, and the real and imaginary parts of the constant B_0 give the strengths of the electric and magnetic fields. At $\alpha = \rho$ this solution describes a uniform, static electromagnetic field, known as Melvin's universe, while at $\alpha = t$ it describes a solution of the cosmological type with a uniform electromagnetic field.

For solution (6.5.1) calculations yield

$$\overset{\circ}{W}=\begin{pmatrix} 4\alpha^2/W^2 & -8ik\alpha^2\beta/W+4i(w-\beta) & -2iB_0\alpha^2/W \\ 8ik\alpha^2\beta/W-4i(w-\beta) & 4(\epsilon W^2+4k\beta^2) & 4B_0\beta \\ 2i\overline{B}_0\alpha^2/W & 4\overline{B}_0\beta & 1 \end{pmatrix},$$

$$\overset{\circ}{\Psi}=\frac{1}{\Sigma}\times$$

$$\begin{pmatrix} W+2\epsilon k\Sigma^2 & -2ik\Sigma(w-\beta) & iB_0\Sigma \\ i\epsilon(w-\beta)/W & \Sigma/W & 0 \\ -2i\epsilon\overline{B}_0[\Sigma^2+(w-\beta)(w-\beta+2\beta/W)] & -4\overline{B}_0\Sigma(w-\beta+\beta/W) & \Sigma \end{pmatrix},$$

$$\overset{\circ}{K}=\begin{pmatrix} -4\epsilon & 0 & 0 \\ 0 & 4\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For a one-soliton solution with this background we have $k = \{1, c, d\}$, $w = i\sigma$,

$$m^A=\begin{pmatrix} Q/W \\ P+4\beta\overline{B}_0S \\ S \end{pmatrix}, \quad n_a=\frac{2i\sigma}{D}\begin{pmatrix} R \\ -[\overline{c}\overline{\Sigma}+i\epsilon(i\sigma+\beta)]/W \end{pmatrix}, \tag{6.5.2}$$

with

$$\begin{aligned}
P &= cW + 4\bar{B}_0 W (i\sigma - \beta)(d - iB_0 Q/2W),\\
Q &= \Sigma - i\epsilon (i\sigma - \beta) c - i\alpha^2 \bar{B}_0 d,\\
R &= W - 4i\epsilon B_0 W \bar{\Sigma} (\bar{d} + i\bar{B}_0 \bar{Q}/2W), \qquad (6.5.3)\\
S &= d - (iB_0/W) Q,\\
D &= \Sigma - (c\bar{c} + 4\epsilon d\bar{d}) \bar{\Sigma} + 2\epsilon\sigma c\\
&\quad + 8\sigma\bar{B}_0 (d - iB_0 Q/2W)(i\sigma + \beta - i\epsilon\bar{c}\bar{\Sigma}).
\end{aligned}$$

We did not write the third component of n_A due to its complexity and because it does not enter into the expression (6.2.16) for the components of the metric and the electromagnetic field.

Just as before, we restrict our discussion to the simplest components of the metric and the electromagnetic field:

$$\begin{aligned}
\Phi_2 &= 2B_0\beta - (4\sigma/D) RS, \qquad (6.5.4)\\
h_{22} &= -\epsilon W^2 + (2\sigma/D) PR + (2\sigma/\bar{D}) \bar{P}\bar{R} + (16\sigma^2/D\bar{D}) R\bar{R}S\bar{S}.
\end{aligned}$$

We start with $\epsilon = 1$ and put $\alpha = \rho$, $\beta = t$, $x^2 = z$, and $x^3 = \varphi$, i.e. a cylindrical wave in an external electrovacuum field. This wave converges to the axis, then is reflected by it, and goes to infinity. We assume for simplicity that in (6.5.3) and (6.5.4) $c = i$.

As $\rho \to 0$, i.e. near the symmetry axis, the field is

$$\begin{aligned}
\Phi_z &= \mathring{\Phi}_z + \sigma/\bar{d}\,(t + i\sigma) + \text{const} + O(\rho^2),\\
h_{zz} &= -1 + O(\rho^2).
\end{aligned}$$

We see that as $t \to \pm\infty$, the perturbation produced by the wave near the axis vanishes, i.e. the wave is localized.

In the stationary case, where $\epsilon = -1$, $\alpha = \rho$, and $\beta = z$, we proceed with the same transformations of coordinates and parameters as for stationary solitons on a flat background. For the quantities in (6.5.3) we have

$$\begin{aligned}
P &= -\frac{2W(m-ib)}{\sigma-a} - 4\bar{B}_0 W [i\sigma - (r-m)\cos\theta] \left(\frac{e}{\sigma-a} + \frac{iB_0 Q}{2W}\right),\\
Q &= r - m - i\sigma\cos\theta - \frac{2i(m-ib)}{\sigma-a} [i\sigma - (r-m)\cos\theta] + \frac{ic\bar{B}_0\rho^2}{\sigma-a},\\
R &= W - 4iB_0 W (r - m + i\sigma\cos\theta) \left(\frac{\bar{e}}{\sigma-a} - \frac{i\bar{B}_0\bar{Q}}{2W}\right),\\
S &= -\frac{e}{\sigma-a} - \frac{iB_0 Q}{W},\\
D &= \frac{2\sigma}{\sigma-a}(r + ia\cos\theta - ib) - 8\sigma\bar{B}_0 \left(\frac{e}{\sigma-a} + \frac{iB_0 Q}{2W}\right)\\
&\times \left[i\sigma + (r-m)\cos\theta - \frac{2i(m+ib)}{\sigma-a}(r - m + i\sigma\cos\theta)\right].
\end{aligned}$$

From this and (6.5.4) it follows that for r's that are small compared to the characteristic scale of the external field, $|B_0|^{-1}$, i.e. for $r \ll |B_0|^{-1}$, the solution (6.5.4) coincides with the Kerr-Newman solution (6.4.4) to within small terms. It is easy to show that at $r \gg m$, σ, a, or $|e|$, and $b = 0$, the solution (6.5.4) combined with (6.5.3) is simply the background solution (6.5.1) for a uniform electromagnetic field, provided we ignore small terms.

Hence, it is natural to assume that (6.5.4) describes the interaction of a field produced by a source of the Kerr-Newman type with an external uniform electromagnetic field.

On the other hand, from the physics of the problem it is evident that a charged source placed in an electric field will be accelerated by this field. However, in the stationary solution that we have just built the source is at rest, which makes the above interpretation inconsistent.

Here we are confronted with a fairly common situation, which consists in the following. Although a formal application of the above algorithm yields a solution that for fixed values of the free parameters coincides with the background (i.e. solitons are absent), the existence of a mathematically rigorous limiting transition to the background solution does not necessarily imply a correct physical interpretation. We must also analyze the global properties of the obtained solution, such as the geodesic completeness and the absence of closed time-like curves and various singularities. The necessity of these properties is dictated by the physics of the problem considered.

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7 Neutrino Fields in General Relativity

N. R. Sibgatullin, *D.Sc. (Physics and Mathematics)*
M. V. Lomonosov Moscow State University

INTRODUCTION

This work is devoted to the interaction of sourceless gravitational and neutrino fields within the scope of general relativity. The stress is on the properties of these fields when an Abelian group of motions G_2 on V_2 is present. In this case it becomes possible to reduce the Maxwell-Einstein-Weyl set of equations for interacting sourceless gravitational, neutrino, and electromagnetic fields to a set of canonical equations. In Sec. 7.1 these equations are derived for the cases of stationary axisymmetric fields and plane waves.

Section 7.2 makes clear that the equations derived in Sec. 7.1 possess remarkable group properties. First we show that for a denumerable set of infinitesimal solutions there exists an infinite dimensional algebra. The algebra is then exponentiated and brought into the form of an infinite set of linear algebraic equations, which proves to be equivalent to a matrix integral equation. We then build an exact solution for any type of neutrino field, which may serve as a starting point for obtaining other types of solutions.

Section 7.3 is devoted to describing new exact solutions of equations derived in Sec. 7.1: in Sec. 7.3.1 we give a class of solutions described by a first-order equation, in Sec. 7.3.2 we study a self-similar collision of gravitational and neutrino waves, while in Sec. 7.3.3 we give the simplest solutions for a stationary neutrino vacuum. Finally, in Sec. 7.3.4 we obtain new exact solutions built from the initial solution for the case where the shift along the orbit of the infinite dimensional algebra is given by an arbitrary rational function of an analytic parameter. These types of solutions in the absence of neutrino include the Sato-Tomimatzu and many other known solutions. For instance, we give the solution for the case of N black holes in arbitrary neutrino fields. For a special type of neutrino fields we were able to build solutions with two arbitrary functions.

In Sec. 7.4 we describe an effect that appears when discontinuities in gravitational fields propagate through neutrino fields: because of the latter the polarization plane of the gravitational waves begins to rotate.

7.1 CANONICAL EQUATIONS OF NEUTRINO FIELDS AND WAVES

For zero rest mass, as the reader must know, the Dirac equation for a spinor field breaks up into two independent equations (Weyl equations) describing neutrino fields.

Let us assume that space is filled by sourceless electromagnetic and neutrino fields, which bring about gravitational fields (the neutrino-electromagnetic vacuum). The total set of equations describing this vacuum consists of the Einstein equations with the energy-momentum tensor of the neutrino and electromagnetic fields, and the Weyl and Maxwell equations in vacuum.

Let us consider the solutions for this set of equations that possess an Abelian group of motions G_2. If both (commutative) Killing vectors are space-like, we are dealing with plane waves, while if one is time-like and the other, the space-like, has closed orbits, we are dealing with stationary fields with axial symmetry. We start with the second case.

(a) We write the interval in the form

$$ds^2 = 2\,(l_i\,dx^i)\,(n_j\,dx^j) - 2\,(m_i\,dx^i)\,(m_j^*\,dx^j) \tag{7.1.1}$$

(* stands for complex conjugation). The vectors **l**, **n**, **m**, and **m*** form a complex tetrad and, by agreement, do not depend on time t or angle φ (in what follows we denote the indices by capital letters A, B, etc.; $A, B = 1,2$). The other coordinates we denote by Greek letters $\mu, \nu \ldots = 3, 4$. An important case is when the matrix of the metric tensor cannot be written in block form, i.e. when there are no two-dimensional surfaces that intersect both Killing vectors orthogonally.

We project the sought set of equations on the above tetrad field, which enables using the calculations of Newman and Penrose [1].

Let D, Δ, δ, δ^* be the operators of differentiation along the vectors **l**, **n**, **m**, **m***.

The Weyl equations for a neutrino field are

$$\begin{aligned} D\Phi + \delta^*\Psi &= (\rho - \varepsilon)\,\Phi + (\alpha - \pi)\,\Psi, \\ \delta\Phi + \Delta\Psi &= (\tau - \beta)\,\Phi + (\gamma - \mu)\,\Psi, \end{aligned} \tag{7.1.2}$$

where Φ and Ψ are the spinor components describing the neutrino field (here we use a gauge that nullifies the other components of the Dirac 4-spinor).

To calculate the rotation coefficients of the complex tetrad, α, β, γ, ε, etc., we restrict the choice of the reference frame by requiring that the metric coefficients $g_{\mu\nu}$ be conformally Euclidean. Next we introduce the complex coordinates $z = (x + iy)/\sqrt{2}$ and

$z^* = (x - iy)/\sqrt{2}$. The interval then takes the form

$$ds^2 = g_{AB}\,(dx^A - \theta g^A\,dz + \theta g^{A*}\,dz^*)\,(dx^B + \theta g^B dz + \theta g^{B*}\,dz^*)$$
$$- 2\theta^2\,dz\,dz^*, \tag{7.1.3}$$

with g_{AB} and θ real-valued functions of x^μ, and the g^A complex-valued functions of x^μ.

We introduce l_A and n_B in the following manner:

$$g_{AB}\,dx^A\,dx^B = 2\,(l_A\,dx^A)\,(n_B\,dx^B).$$

The capital Latin indices can be raised or lowered by applying the metric coefficients g_{AB}. Then, if we apply the differentiation operations D and Δ to scalar quantities describing fields, we obtain zero.

Suppose that $r \equiv (|\det g_{AB}|)^{1/2}$. Then from the definition of tetrad rotation coefficients we have

$$\begin{aligned}
&\tau = -\pi^* = -\frac{1}{2\theta r}\,\frac{\partial r}{\partial z};, \quad k = -\frac{1}{\theta}\,l^A\,\frac{\partial l_A}{\partial z}, \quad \nu = \frac{1}{\theta}\,n^A\,\frac{\partial n_A}{\partial z},\\
&\rho = -\rho^* = 2\varepsilon = \frac{1}{2\theta^2}\,l_B\left[\frac{\partial(\theta g^B)}{\partial z^*} - \frac{\partial(\theta g^{B*})}{\partial z}\right],\\
&\mu = -\mu^* = 2\gamma = \frac{1}{2\theta^2}\,n_B\left[\frac{\partial(\theta g^B)}{\partial z^*} - \frac{\partial(\theta g^{B*})}{\partial z}\right],\\
&\alpha = \frac{1}{4\theta}\left[n^A\,\frac{\partial l_A}{\partial z^*} - l^A\,\frac{\partial n_A}{\partial z^*}\right] - \frac{1}{2\theta^2}\,\frac{\partial\theta}{\partial z^*},\\
&\beta = \frac{1}{4\theta}\left[n^A\,\frac{\partial l_A}{\partial z} - l^A\,\frac{\partial n_A}{\partial z}\right] + \frac{1}{2\theta^2}\,\frac{\partial\theta}{\partial z},\\
&\sigma = \lambda = 0.
\end{aligned} \tag{7.1.4}$$

The components $4\pi T_{ij}l^i m^j$ and $4\pi T_{ij}n^i m^j$ of the energy-momentum tensor of the neutrino field can be expressed in terms of the spinor field components Φ and Ψ in the following manner[1]:

$$\Phi_{01} = 4\pi i\,[\Psi\delta\Psi^* - \Psi^*\delta\Psi - k\Phi\Phi^* + \rho\Phi^*\Psi$$
$$+ (\beta - \alpha^* - \pi^*)\,\Psi\Psi^*],$$
$$\Phi_{12} = 4\pi i\,[\Phi\delta\Phi^* - \Phi^*\delta\Phi - \nu^*\Psi\Psi^* - \mu\Psi^*\Phi$$
$$+ (\alpha^* - \beta - \tau)\,\Phi\Phi^*]$$

(here we use a system of units in which the speed of light and the gravitational constant are equal to unity).

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[1] The respective energy-momentum tensor components for an electromagnetic field are zero.

According to the Einstein equations, the corresponding Ricci tensor components are

$$\Phi_{01} = (\delta - \alpha^* - \beta - 3\tau)\rho/2 + k\mu/2,$$

$$\Phi_{12} = (\delta + \alpha^* + \beta - 3\tau)\mu/2 + \nu^*\rho/2.$$

What is important for subsequent reasoning is the fact that these field equations (combined with Weyl equations (7.1.2)) can be written in the form of a set of equations

$$(\delta - \alpha^* - \beta - 3\tau) A - kB = 0, \qquad (7.1.5)$$
$$(\delta + \alpha^* + \beta - 3\tau) B + \nu^* A = 0,$$
$$A \equiv \rho + 8\pi i\Psi\Psi^*, \quad B \equiv \mu - 8\pi i\Phi\Phi^*.$$

Next we use only trivial solutions of Eqs. (7.1.5), i.e.

$$A = B = 0. \qquad (7.1.6)$$

Then Weyl equations (7.1.2) can be written thus:

$$\delta^*\Psi = -4\pi i\Psi^*\Psi\Phi + (\alpha - \pi)\Psi, \qquad (7.1.2')$$
$$\delta\Phi = -4\pi i\Phi\Phi^*\Psi + (\tau - \beta)\Phi.$$

If we combine Eqs. (7.1.4) for α, β, τ, and π with (7.1.2′), we have an important first integral

$$\Phi\Psi^* = [i/(8\pi\theta r)]\, \partial w/\partial z^*, \qquad (7.1.7)$$

where w is an arbitrary real-valued harmonic function, and the factor $i/8\pi$ is introduced for convenience. The arbitrary analytic function is written in the form of a derivative of w.

Let us now consider the components $\Phi_{00} = 4\pi T^{(\nu)}_{ij} l^i l^j$ and $\Phi_{22} = 4\pi T^{(\nu)}_{ij} n^i n^j$ of the energy-momentum tensor of the neutrino field:

$$\Phi_{00} = 8\pi i\,[k\Phi\Psi^* - k^*\Psi\Phi^* + \underline{(\varepsilon - \varepsilon^*)\Psi\Psi^*}], \qquad (7.1.8)$$
$$\Phi_{22} = 8\pi i\,[\nu^*\Phi\Psi^* - \nu\Psi\Phi^* + \underline{(\gamma^* - \gamma)\Phi\Phi^*}].$$

The underlined terms in (7.1.8) cancel out with the corresponding terms $-\rho^2$ and $-\mu^2$ in the Ricci tensor. The respective components of the Einstein equations (after we multiply them into θ) contain only components of the energy-momentum tensor of the electromagnetic field, the metric coefficients g_{AB}, and the harmonic function w.

Before writing these equations explicitly, let us consider the following component of the Einstein equations:

$$4\pi T_{ij} l^i n^j = \Phi_{11} = -\rho\mu - (\delta - 2\tau + \beta - \alpha^*)\pi.$$

The electromagnetic part of this energy-momentum tensor component vanishes (in the stationary axisymmetric case the electromagnetic field is described by two complex Newman-Penrose scalars Φ_0 and Φ_2 since $\Phi_1 = 0$). The neutrino part of this component is

$$4\pi i\,[(\varepsilon^* - \varepsilon)\,\Phi\Phi^* + (\gamma - \gamma^*)\,\Psi\Psi^*] = -\rho\mu.$$

Whence the Einstein equations yield the equality

$$(\delta - 2\tau - \alpha^* + \beta)\,\pi = 0, \tag{7.1.9}$$

which when combined with (7.1.4) yields $\partial^2 r/\partial z\,\partial z^* = 0$. Therefore, the real-valued function $r \equiv (|\det g_{AB}|)^{1/2}$ is harmonic. We choose the coordinates z and z^* to within a harmonic mapping $z' = f(z)$, with $f(z)$ an analytic function. To remove this freedom of choice we can, for instance, require that the harmonic function r be equal to the real part of $z\sqrt{2}$.

For the subsequent discussion it is convenient to go over to tensor notation in Eqs. (7.1.8) and (7.1.9). Substituting (7.1.4) into the Einstein equations, we find that

$$\Phi_{00} = 4\pi\,(T_{ij}^{(\nu)} + T_{ij}^{(\mathrm{em})})\,l^i l^j,$$

$$\Phi_{22} = 4\pi\,(T_{ij}^{(\nu)} + T_{ij}^{(\mathrm{em})})\,n^i n^j,$$

$$\Phi_{11} = 4\pi\,(T_{ij}^{(\nu)} + \qquad)\,l^i n^j.$$

Here the indices (ν) and (em) stand for the neutrino and electromagnetic components of the energy-momentum tensor, respectively.

The symmetric two-dimensional second-rank tensor M_{AB} is determined by its three projections $M_{AB}l^Al^B$, $M_{AB}n^An^B$, and $M_{AB}l^An^B$, with the determinant of $g_{AB} = l_An_B + n_Al_B$ being fixed to within an arbitrary transformation from the matrix group $SL\,(2, R)$. This transformation group leaves the Levi-Civita symbols ε_{AB} and ε^{AB} invariant: $\varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon_{AA} = \varepsilon^{AA} = 0$; $A = 1, 2$.

We can show that for a fixed orientation of $l_1n_2 - n_1l_2 = r$ we have the following relationships:

$$rn^A = \varepsilon^{AC}n_C, \quad rl^A = \varepsilon^{AC}l_A, \quad n_A = r\varepsilon_{AC}n^C, \quad l_A = r\varepsilon_{AC}l^C. \tag{7.1.10}$$

We can easily verify that the symmetric tensor with projections

$$\Phi_{00} + \rho^2 = -(\delta^* - 3\alpha - \beta^* + \pi)\,k + \tau k^*,$$

$$\Phi_{11} + \rho\mu = -(\delta - 2\tau + \beta - \alpha^*)\,\pi,$$

$$\Phi_{22} + \mu^2 = (\delta + 3\beta + \alpha^* - \tau)\,\nu + \pi\nu^*$$

is the tensor R_{AB}:

$$R_{AB} \equiv \frac{1}{2}\left[\frac{\partial^2 g_{AB}}{\partial z\partial z^*} + \frac{1}{2r}\frac{\partial r}{\partial z}\frac{\partial g_{AB}}{\partial z^*} + \frac{1}{2r}\frac{\partial r}{\partial z^*}\frac{\partial g_{AB}}{\partial z}\right.$$
$$\left. - \frac{1}{2}\frac{\partial g_{AC}}{\partial z}\frac{\partial g_{BD}}{\partial z^*}g^{CD} - \frac{1}{2}\frac{\partial g_{AC}}{\partial z^*}\frac{\partial g_{BD}}{\partial z}g^{CD}\right]. \tag{7.1.11}$$

According to Eqs. (7.1.7) and (7.1.8), to build the energy-momentum of the neutrino field by three projections, it suffices to build a symmetric tensor χ_{AB} by the following projections:

$$l^A l^B \chi_{AB} = k = -(1/\theta)\, l^A \partial l_A/\partial z, \quad n^A n^B \chi_{AB} = (1/\theta)\, n^A \partial n_A/\partial z.$$

We can easily verify that, if we account for (7.1.10), the symmetric tensor is

$$\chi_{AB} = \frac{1}{4\theta r}\left[g_{AC} \frac{\partial g^{CD} r}{\partial z} \varepsilon_{BD} + g_{BC} \frac{\partial g^{CD} r}{\partial z} \varepsilon_{AD} \right].$$

From the Einstein equations we obtain

$$R_{AB} = -(\chi_{AB} \partial w/\partial z^* + \chi^*_{AB} \partial w/\partial z)\, r + T^{(\mathrm{em})}_{AB} \theta^2 r,$$

where $T^{(\mathrm{em})}_{AB}$ are the corresponding energy-momentum tensor components of the electromagnetic field.

Contracting both sides with g^{BC}, we can transform the equations to the following form:

$$\frac{\partial}{\partial z}\left[r g^{BC} \frac{\partial (g_{AC} + w\varepsilon_{AC})}{\partial z^*} \right] + \frac{\partial}{\partial z^*}\left[r g^{BC} \frac{\partial (g_{AC} + w\varepsilon_{AC})}{\partial z} \right]$$
$$= 16\pi\theta^2 r T^{(\mathrm{em})}_{AB} g^{BC}. \qquad (7.1.12)$$

In the stationary axisymmetric case, the only components of the electromagnetic field 4-potential A_i that are nonzero are A_B.

The Maxwell equations in vacuum yield equations for A_C:

$$\nabla_i F^{iB} = \frac{1}{\sqrt{-\tilde{g}}} \frac{\partial}{\partial x^\mu}\left[\sqrt{\tilde{g}}\, (\tilde{g}^{\mu\nu} \tilde{g}^{BC} - \tilde{g}^{\mu C} \tilde{g}^{B\nu}) \frac{\partial A_C}{\partial x^\nu} \right] = 0. \qquad (7.1.13)$$

The tildes over the contravariant components of the metric tensor imply that the components are calculated as the components of the inverse to matrix g_{ij}. In the metric (7.1.3), $\nabla_i F^{i\mu} = 0$ implies, as we can easily verify, that

$$r\Phi_1 = r\, (g^{B*}\, \partial A_B/\partial z^* - g^B\, \partial A_B/\partial z) = \mathrm{const} = C_1. \qquad (7.1.14)$$

In the particular case where $C_1 = 0$, the Newman-Penrose scalar Φ_1 vanishes, a fact we used earlier.

For a more vivid picture of the invariance of the equations in relation to the $SL\,(2, R)$ group, in what follows we will raise and lower the indices via tensor ε_{AB} [2]:

$$f_{AB} = g_{AB} = f_A{}^C \varepsilon^{BC}, \quad f^{AB} = f_{DC} \varepsilon^{BC} \varepsilon^{AD} = -g^{AB} r^2, \quad f_A{}^B f_{BC} = r^2 \varepsilon_{AC}.$$

If we take into account (7.1.14), we can easily transform Eq. (7.1.13) to the following form:

$$\frac{\partial}{\partial z}\left[\frac{1}{r} f_B{}^C \frac{\partial A_C}{\partial z^*} \right] + \frac{\partial}{\partial z^*}\left[\frac{1}{r} f_B{}^C \frac{\partial A_C}{\partial z} \right] = 0. \qquad (7.1.15)$$

We go over to the real variables x and y and introduce the operator $\nabla = (\partial/\partial x,\ \partial/\partial y)$, by which we mean the gradient when we apply the operator to a scalar or the divergence when we apply it to a vector. Equation (7.1.15) can then be written thus:

$$\nabla\,(r^{-1}f_A{}^C\nabla A_C) = 0. \qquad (7.1.15')$$

This means that there exist potentials B_A such that

$$r^{-1}f_A{}^C\,\nabla A_C = \tilde{\nabla} B_A, \quad \text{where} \quad \tilde{\nabla} \equiv (\partial/\partial y,\ -\partial/\partial x). \qquad (7.1.16)$$

We introduce, following Kinnersley [3], the complex vector potential $\Phi_A \equiv A_A + iB_A$. Then it is easy to see that the new potential satisfies the equation

$$ir\,\nabla\Phi_A = f_A{}^B\,\tilde{\nabla}\Phi_B. \qquad (7.1.17)$$

We now express the energy-momentum tensor components of the electromagnetic field, $T^{(\mathrm{em})}_{AB}\,g^{BC}$, in the metric (7.1.3) and arrive at a closed set of equations for the neutrino-electromagnetic vacuum. Thus

$$4\pi T^{(\mathrm{em})}_{AB} = -F_{\mu A}F^{\mu}{}_B + (1/2)\,g_{AB}F_{\mu C}F^{\mu C}$$
$$= \frac{1}{\theta^2}\left[\frac{\partial A_E}{\partial z}\frac{\partial A_D}{\partial z^*} + \frac{\partial A_E}{\partial z^*}\frac{\partial A_D}{\partial z}\right][\delta^E_A\delta^D_B - (1/2)\,g^{ED}g_{AB}].$$

Multiplying this formula into g^{BC}/θ^2 (not to be confused with $\tilde{g}^{BC}/\theta^2$), we have

$$4\pi\theta^2 r T^{C(\mathrm{em})}_A = r\left[\frac{\partial A_E}{\partial z}\frac{\partial A_D}{\partial z^*} + \frac{\partial A_E}{\partial z^*}\frac{\partial A_D}{\partial z}\right]\left[\delta^E_A g^{DC} - \frac{1}{2}\,g^{DE}\delta^C_A\right].$$

Using the field equations (7.1.15), we write the right-hand side in the form of the divergence of a vector:

$$\nabla\,\{r\,[\delta^E_A g^{DC} - (1/2)\,g^{ED}\delta^C_A]\,A_E\nabla A_D\}.$$

If we now use the equality $\delta^C_A f^{ED}A_E\nabla A_D \equiv A_A f^{CD}\nabla A_D - A^C f_A{}^D\,\nabla A_D$, we can rewrite the above expression as

$$\nabla\,\{[1/(2r)]\,[A_A f^{DC}\nabla A_D + A^C f_A{}^D\nabla A_A]\}.$$

The Einstein equations take the form

$$\nabla\,\{(1/r)\ f_A{}^B\nabla\ (f_B{}^C + w\delta^C_B) - 2A_A f^{DC}\nabla A_D - 2f_A{}^D A^C\nabla A_D\} = 0. \quad (7.1.18)$$

This implies that there is a matrix potential function

$$r\tilde{\nabla}\Psi_A{}^C = f_A{}^B\nabla\,(f_B{}^C + w\delta^C_B) - 2A_A f^{DC}\nabla A_D - 2f_A{}^D A^C\nabla A_D.$$

Let us now introduce a complex matrix potential $H_A{}^B$:

$$\nabla H_A{}^B = \nabla (f_A{}^B + w\delta_A^B) - \Phi^{B*}\nabla\Phi_A$$
$$- (i/r) f_A{}^C [\tilde{\nabla} (f_C{}^B + w\delta_C^B) - \Phi^{B*}\nabla\Phi_C]. \tag{7.1.19}$$

Such a potential does exist if we assume that Eq. (7.1.18) is valid together with (7.1.16) and (7.1.17).

Formula (7.1.19) implies that the matrix potential $H_A{}^B$ satisfies

$$ir \nabla H_A{}^B = f_A{}^C \tilde{\nabla} H_C{}^B. \tag{7.1.20}$$

Following Kinnersley [3], we introduce the potentials L^B and K:

$$\nabla L^B = 2\Phi^{*C} \nabla H_C{}^B, \quad \nabla K = 2\Phi^{C*} \nabla\Phi_C.$$

These potentials exist provided $\tilde{\nabla}\Phi^{C*} \cdot \nabla H_C{}^B = 0$, $\tilde{\nabla}\Phi^{C*} \cdot \nabla\Phi_C = 0$, which are valid due to Eqs. (7.1.17) and (7.1.20).[2]

Following Hauser and Ernst [4], we build the complex matrix $H_a{}^b$ in the following way:

$$H_a{}^b = \begin{pmatrix} H_A{}^B & \Phi_A \\ L^B & K \end{pmatrix}, \quad \text{i.e.} \quad H_3{}^B = L^B, \quad H_A{}^3 = \Phi_A, \quad H_3{}^3 = K.$$

Note that from (7.1.17), (7.1.19), and (7.1.20) follows

$$2ir \nabla^2 H_a{}^b = \nabla H_a{}^C \tilde{\nabla} H_c{}^b; \quad a, b, c = 1, 2, 3, \tag{7.1.21}$$

which implies that $\nabla^2 H_a{}^a = 0$. The definition of $H_a{}^b$ and (7.1.19) imply that $\nabla H_a{}^a = 2 (\nabla w - i \nabla r)$. Let us introduce a harmonic function q conjugate to the harmonic function r, i.e. $\tilde{\nabla} r = -\nabla q$. Then the trace of matrix $H_a{}^b$ is

$$H_a{}^a = 2 (w + iq). \tag{7.1.22}$$

Take the components $H_1{}^2$ and $H_1{}^3$ of Eq. (7.1.21). Combining (7.1.17), (7.1.20), and (7.1.22), from (7.1.21) we obtain a set of equations closed with respect to these components:

$$\begin{aligned} &(\Phi\Phi^* + \mathrm{Re}\, H) \left[\nabla^2 + \frac{\nabla (r + i\tilde{w})}{r} \nabla\right] \Phi = \nabla\Phi (\nabla H + 2\Phi^* \nabla\Phi), \\ &(\Phi\Phi^* + \mathrm{Re}\, H) \left[\nabla^2 + \frac{\nabla (r + i\tilde{w})}{r} \nabla\right] H = \nabla H (\nabla H + 2\Phi^* \nabla\Phi), \end{aligned} \tag{7.1.23}$$

$$H \equiv -H_1{}^2, \quad \Phi \equiv \Phi_1, \quad \nabla\tilde{w} = \tilde{\nabla} w.$$

The function $\tilde{w}$ is the harmonic conjugate of w.

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[2] It is worth noting that Eqs. (7.1.17.) and (7.1.20) together with the definitions of L^B and K can be written in a form of a single 3×3 matrix equation for matrix $H_a{}^b$ (a, b—1, 2, 3): $(-2Z\varepsilon + \varepsilon H - H^+\varepsilon - \Pi/2) H,{}_z = 0$, where $+$ denotes hermitian conjugation, $z = \rho + iq$, $\varepsilon^{ab} = \varepsilon^{AB}$, $a = A$, $b = B$, $\Pi^{33} = 1$, the other components of ε and Π are zero.

(b) Now we turn to the case where both commutative Killing vectors are space-like (the case of plane waves). In such Riemannian spaces the interval can be written as

$$dS^2 = 2\theta^2\, du dv - g_{AB}\, (dx^A + g_u^A\, du + g_v^A\, dv)$$
$$\times\, (dx^B + g_u^B du + g_v^B dv),$$

where the functions θ, g_u^B, g_v^B, and g_{AB} are the sought functions of the retarded and advanced times u and v, respectively. In this case the two-dimensional metric g_{AB} is positive definite and can be written thus:

$$g_{AB} = m_A m_B^* + m_B m_A^*, \quad m_A m^A = m_A^* m^{*A}, \quad m_A m^{*A} = 1$$

(the lowering and raising of indices is achieved by applying the matrix g_{AB}). In this case it is natural to define the field of isotropic tetrads in the following way:

$$m_i\, (\theta m_A g_u^A,\ \theta m_A g_v^A,\ -m_1,\ -m_2), \quad l_i\, (\theta,\ 0,\ 0,\ 0), \quad n_i\, (0,\ \theta,\ 0,\ 0),$$
$$m^i\, (0,\ 0,\ m^1,\ m^2), \quad l^i\, (0,\ \theta^{-1},\ g_v^1,\ g_v^2), \quad n^i\, (\theta^{-1},\ 0,\ g_u^1,\ g_u^2).$$

Applying the operators δ and δ^* to the various scalar characteristics of the fields that we are studying yields zero. For the tensor characteristics we must nullify the respective Lie derivatives.

Now suppose that $r = (\det g_{AB})^{1/2}$. For this case the coefficients of tetrad rotation are

$$\tau = \pi^* = 2\beta = 2\alpha^* = \frac{m^A}{\theta}\left(\frac{\partial g_{Aj}}{\partial v}\, n^j - \frac{\partial g_{Aj}}{\partial u}\, l^j\right),$$

$$\rho = -\frac{1}{2r\theta}\,\frac{\partial r}{\partial v}, \quad \sigma = -\frac{1}{\theta}\, m^A\, \frac{\partial m_A}{\partial v}, \quad \mu = \frac{1}{2r\theta}\,\frac{\partial r}{\partial u},$$

$$\lambda = \frac{1}{\theta}\, m^{A*}\, \frac{\partial m_A^*}{\partial u}, \quad \varepsilon = \frac{1}{2}\left[\frac{1}{\theta^2}\,\frac{\partial\theta}{\partial v} + \frac{1}{20}\, m^{A*}\,\frac{\partial m_A}{\partial v} - \frac{1}{20}\, m^A\, \frac{\partial m_A^*}{\partial v}\right],$$

$$\gamma = \frac{1}{2}\left[-\frac{1}{\theta^2}\,\frac{\partial\theta}{\partial u} + \frac{1}{2\theta}\, m^{A*}\,\frac{\partial m_A}{\partial u} - \frac{1}{2\theta}\, m^A\, \frac{\partial m_A^*}{\partial u}\right], \quad k = \nu = 0.$$

Curiously, the Einstein equations for the components Φ_{10} and Φ_{21}, to which the electromagnetic field contributes nothing, can be written, respectively, in the following form:

$$(D - 3\rho + \varepsilon - \varepsilon^*)\, (\alpha + 4\pi i\Phi\Psi^*) - \sigma^*\, (\alpha^* - 4\pi i\Phi^*\Psi) = 0,$$

$$(\Delta + 3\mu + \gamma - \gamma^*)\, (\alpha + 4\pi i\Phi\Psi^*) + \lambda\, (\alpha^* - 4\pi i\Phi^*\Psi) = 0.$$

This set of equations overdetermined with respect to $\alpha + 4\pi i\Phi\Psi^*$ has a trivial solution

$$\alpha + 4\pi i\Phi\Psi^* = 0. \qquad (7.1.24)$$

The Weyl equations then take the form

$$D\Phi = (\rho - \varepsilon)\,\Phi + 4\pi i \Phi \Psi^* \Psi,$$

$$\Delta\Psi = -4\pi \Psi \Phi \Phi^* + (\gamma - \mu)\,\Psi.$$

These equations yield the following first integrals

$$\Phi\Phi^* = c_1(u)/8\pi r\theta, \quad \Psi\Psi^* = c_2(v)/8\pi r\theta,$$

where c_1 and c_2 are arbitrary functions of u and v, respectively.

The component Φ_{20} of the energy-momentum tensor of the neutrino field has the following form:

$$\Phi_{20} = 8\pi i\,[\sigma^*\Phi\Phi^* + 2\alpha\Phi\Psi^* + \lambda\Psi\Psi^*]. \tag{7.1.25}$$

The respective component of the Einstein tensor is expressed thus:

$$\left(\frac{1}{\theta}\frac{\partial}{\partial v} - \frac{1}{2r\theta}\frac{\partial r}{\partial v} + \frac{1}{\theta^2}\frac{\partial\theta}{\partial v} + \frac{1}{2\theta} m^{A*}\frac{\partial m_A}{\partial v} - \frac{1}{2\theta} m^A \frac{\partial m_A^*}{\partial v}\right)\lambda$$
$$-4\alpha^2 + \frac{1}{2r\theta}\frac{\partial r}{\partial u} m^{A*}\frac{\partial m_A^*}{\partial v}.$$

Note that after we multiply the respective Einstein equation into θ^2, it contains only the components of tensor g_{AB}, the functions $c_1(u)$ and $c_2(v)$ (characterizing the neutrino field), and the components of the electromagnetic field tensor, since in view of Eq. (7.1.25) the terms with α cancel out.

The component Φ_{11} of the Ricci tensor is

$$(1/r\theta^2)\,\partial^2 r/\partial u \partial v - 4\alpha\alpha^*. \tag{7.1.26}$$

On the other hand, the similar component of the energy-momentum tensor of the neutrino field is

$$8\pi i\,(\alpha^*\Phi\Psi^* - \alpha\Phi^*\Psi).$$

Bearing in mind (7.1.26), we obtain a wave equation for the function $r \equiv (\det g_{AB})^{1/2}$, whence $r = f(u) + \Psi(v)$.

We write the expression $\Phi_{20} + 4\alpha^2$ in the form $m^{A*}m^{B*}R_{AB}$, with R_{AB} a symmetric second-rank tensor. This tensor can then be built for a fixed r to within a transformation from the $SL(2, R)$ group:

$$x^{1'} = Ax^1 + Bx^2, \quad x^{2'} = Cx^1 + Dx^2, \quad AD - CB = 1.$$

We do the same with the component $\Phi_{20} + 4\alpha^2$ of the neutrino field:

$$r\theta^2 4\,(\Phi_{20} + 4\alpha^2) = m^{A*}m^{B*}\left[c_1(u)\left(g_{AD}\frac{\partial(g^{DC}r)}{\partial v}\varepsilon_{CB} + g_{BD}\frac{\partial(g^{DC}r)}{\partial v}\varepsilon_{CA}\right)\right.$$
$$\left.-\,c_2(v)\left(g_{AD}\frac{\partial(g^{DC}r)}{\partial u}\varepsilon_{CD} + g_{BD}\frac{\partial(g^{DC}r)}{\partial u}\varepsilon_{CA}\right)\right].$$

(Here we used the fact that $rm^{B*}\varepsilon_{CB} = im_C^*$ and $m_1m_2^* - m_2m_1^* = ir$.) After this the procedure is exactly the same as in case (a), and for this reason we give only the final result:

$$[rg^{AC}(g_{BC}+C\varepsilon_{BC})_{,u}]_{,v}+[rg^{AC}(g_{BC}+C\varepsilon_{BC})_{,v}]_{,u}$$
$$=16\pi g^{AC}T_{BC}^{(\mathrm{em})}\theta^2 r, \quad C_{,u}=c_1(u), \quad C_{,v}=c_2(v), \tag{7.1.27}$$

where $T_{BC}^{(\mathrm{em})}$ is again the corresponding energy-momentum tensor components of the electromagnetic field.

Suppose that A_B are the nonzero components of the 4-vector potential of the electromagnetic field. The Maxwell equations in vacuum are

$$(rg^{AB}A_{B,u})_{,v}+(rg^{AB}A_{B,v})_{,u}=0. \tag{7.1.28}$$

In what follows the raising and lowering of indices is done by applying the tensor ε_{AB}.

If we calculate the energy-momentum tensor of the electromagnetic field and substitute the result into (7.1.27), we obtain

$$\{r^{-1}[f_A{}^C(f_C{}^B+C\delta_C^B)_{,u}-2f_A{}^CA_{C,u}A^B-2f^{BC}A_{C,u}A_A]\}_{,v}$$
$$+\{r^{-1}[f_A{}^C(f_C{}^B+C\delta_C^B)_{,v}-2f_A{}^CA_{C,v}A^B-2f^{BC}A_{C,v}A_A]\}_{,u}=0, \tag{7.1.29}$$
$$f_A{}^Cf_C{}^B=-r^2\delta_A^B.$$

If to this matrix equation we add Eq. (7.1.28), which we write as

$$(r^{-1}f_A{}^CA_{C,u})_{,v}+(r^{-1}f_A{}^CA_{C,v})_{,u}=0, \tag{7.1.28'}$$

we obtain a closed set of equations. Next we reintroduce the variables x and t by the formulas $u = (x + t)/\sqrt{2}$ and $v = (x - t)/\sqrt{2}$. Then, in view of the divergent form of Eq. (7.1.28′), there exists a potential vector function B_A:

$$r^{-1}f_A{}^CA_{C,x}=B_{A,t}, \quad r^{-1}f_A{}^CA_{C,t}=B_{A,x}. \tag{7.1.30}$$

If we build the linear form $A_B + iB_B$, Eq. (7.1.30) yields a self-duality equation for the gradients of Φ_A:

$$r\nabla\Phi_A = if_A{}^C\tilde{\nabla}\Phi_C, \quad \text{where} \quad \nabla = (\partial/\partial t,\ \partial/\partial x), \quad \tilde{\nabla} = (\partial/\partial x,\ \partial/\partial t). \tag{7.1.31}$$

The matrix equation (7.1.29) implies that there exists a complex matrix function $H_A{}^B$ such that

$$\nabla H_A{}^B=\nabla(f_A{}^B+C\delta_A^B)-\Phi^{B*}\nabla\Phi_A$$
$$+(i/r)[\tilde{\nabla}(f_C{}^B+C\delta_C^B)-\Phi^{B*}\tilde{\nabla}\Phi_C]f_A{}^C \tag{7.1.32}$$

From the definition of $H_A{}^B$ it follows that

$$r\nabla H_A{}^B=if_A{}^C\tilde{\nabla}H_C{}^B. \tag{7.1.33}$$

Taking the divergence of both sides of Eqs. (7.1.31) and (7.1.33), we arrive at the following equations:

$$2ir\eta^{ik}\frac{\partial^2\Phi_A}{\partial x^i\,\partial x^k}=\varepsilon^{ik}\frac{\partial\Phi_C}{\partial x^i}\left(\frac{\partial H_A{}^C}{\partial x^k}+2\Phi^{*C}\frac{\partial\Phi_A}{\partial x^k}\right),\quad \|\eta^{ik}\|=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{7.1.34}$$

$$2ir\eta^{ik}\frac{\partial^2 H_A{}^B}{\partial x^i\,\partial x^k}=\varepsilon^{ik}\frac{\partial H_C{}^B}{\partial x^i}\left(\frac{\partial H_A{}^C}{\partial x^k}+2\Phi^{*C}\frac{\partial\Phi_A}{\partial x^k}\right),\quad \|\varepsilon^{ik}\|=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.\tag{7.1.35}$$

Following Hauser and Ernst [4], we introduce the three-dimensional matrix $H_a{}^b$ ($a, b = 1, 2, 3$) such that

$$H_a{}^b = H_A{}^B \quad \text{at} \quad a = A = 1, 2, \quad b = B = 1, 2, \quad H_A{}^3 = \Phi_A,$$

$$\nabla H_3{}^B = 2\Phi^{C*}\,\nabla H_C{}^B, \quad \nabla H_3{}^3 = 2\Phi^{C*}\,\nabla\Phi_C.$$

Combining this with Eqs. (7.1.34) and (7.1.35), we have an equation for $H_a{}^b$:

$$\eta^{ik}\frac{\partial^2 H_a{}^b}{\partial x^i\,\partial x^k}=\frac{i}{2r}\varepsilon^{ik}\frac{\partial H_a{}^c}{\partial x^i}\frac{\partial H_c{}^b}{\partial x^k}.\tag{7.1.36}$$

From the definitions of $H_a{}^b$ and $H_A{}^B$ (the latter is defined in (7.1.32)) it follows that $H_a{}^a = 2(C + iz)$, where z can be found from the equation $\tilde{\nabla} r = -\nabla z$.

For the components $H_1{}^2 = -H$ and $H_1{}^3 = \Phi$ from Eqs. (7.1.31), (7.1.33)-(7.1.35) we find a closed set of equations

$$(\Phi\Phi^* + \operatorname{Re} H)\,\eta^{ik}\times\left[\frac{\partial^2 H}{\partial x^i\,\partial x^k}+\frac{1}{r}\frac{\partial(r+i\tilde{C})}{\partial x^i}\frac{\partial H}{\partial x^k}\right]-\frac{\partial H}{\partial x^i}\left(\frac{\partial H}{\partial x^k}+\Phi^*\frac{\partial\Phi}{\partial x^k}\right)=0,$$

$$(\Phi\Phi^* + \operatorname{Re} H)\,\eta^{ik}\times\left[\frac{\partial^2 \Phi}{\partial x^i\,\partial x^k}+\frac{1}{r}\frac{\partial(r+i\tilde{C})}{\partial x^i}\frac{\partial \Phi}{\partial x^k}\right]-\frac{\partial \Phi}{\partial x^i}\left(\frac{\partial H}{\partial x^k}+\Phi^*\frac{\partial\Phi}{\partial x^k}\right)=0,\tag{7.1.37}$$

where $\tilde{C}$ is determined from the equation $\tilde{\nabla} C = \nabla\tilde{C}$.

The canonical equations of the neutrino electrovacuum obtained here are generalizations of similar electrovacuum equations of Geroch [2], Kinnersley [3], Hauser and Ernst [4], and Kinnersley and Chitre [5] to the case of the presence of a neutrino field.

ON THE INFINITE DIMENSIONAL ALGEBRA AND THE LIE GROUP OF NEUTRINO VACUUM EQUATIONS

Let us consider the invariant transformations of Eqs. (7.1.21) and (7.1.36) continuously depending on parameters and including the identity transformation if the N-dimensional case (the case $N = 2$ corresponds to the gravitational-neutrino vacuum, while $N = 3$ to the case of neutrino electrovacuum).

We introduce the "right" matrix H'_2 in the following manner:

$$\nabla H'_2 = (\nabla H)\, H + 2i\, (q\, \nabla H - r\, \tilde{\nabla} H). \tag{7.2.1}$$

This matrix exists if the following equation for H is valid:

$$2ir\, \nabla^2 H = \nabla H\, \tilde{\nabla} H. \tag{7.2.2}$$

From (7.2.1) it follows that H'_2 satisfies the equation

$$2ir\, \nabla^2 H'_2 = \nabla H\, \tilde{\nabla} H'_2. \tag{7.2.1'}$$

Next we introduce the right matrices H'_n $(n = 2, 3, \ldots)$ by the recurrence formulas

$$\nabla H'_n = (\nabla H)\, H'_{n-1} + 2i\, (q\, \nabla H_{n-1} - r\, \tilde{\nabla} H_{n-1}). \tag{7.2.3}$$

These matrices exist provided that the equations

$$2ir\, \nabla^2 H'_{n-1} = \nabla H\, \tilde{\nabla} H'_{n-1} \tag{7.2.3'}$$

are valid, which we assume is true.

In a similar way we can introduce left matrices $'H_m$ $(m = 2, 3, \ldots)$ by the recurrence formulas

$$\nabla' H_m = {}'H_{m-1}\, \nabla H - 2i\, (q\, \nabla' H_{m-1} - r\, \tilde{\nabla}' H_{m-1}). \tag{7.2.4}$$

These matrices exist provided that the equations

$$2ir\, \nabla^{2\prime} H_{m-1} = \nabla' H_{m-1}\, \tilde{\nabla} H \tag{7.2.4'}$$

are valid, which we assume is true (when $m = 2$, this equation coincides with (7.2.1), and $'H_1 = H'_1 = H$).

Starting from the definitions for $'H_n$ and H'_m, we can show that

$$\nabla \mathcal{H}_n = 0, \quad \mathcal{H}_n \equiv {}'H_n - {}'H_{n-1} H + {}'H_{n-2} H'_2 + \ldots + (-1)^n H'_n. \tag{7.2.5}$$

(From now on we will assume that $\mathcal{H}_n = 0$, $n = 1, 2, \ldots$.) We build the matrix

$$h_n^{\Gamma} \equiv \Gamma' H_n - H \Gamma' H_{n-1} + H'_2 \Gamma' H_{n-2} + \ldots + (-1)^n H'_n, \tag{7.2.6}$$

with Γ an arbitrary constant matrix. By induction we can show that the h_n satisfy the equations

$$2ir\ \nabla^2 h_n = \nabla h_n\ \tilde{\nabla} H + \nabla H\ \tilde{\nabla} h_n. \tag{7.2.7}$$

At $n = 1$ the matrix h_1 forms the solutions $\delta H = \Gamma H - H\Gamma$. But Eqs. (7.2.7) coincide with the equation for small perturbations of Eq. (7.2.2). By the known solution H, then, formulas (7.2.6) yield a denumerable set of solutions depending continuously on the components of matrix Γ and differing from the given solution H by an infinitesimal quantity.

Let us now show that the found set of solutions of the linearized equations for H (7.2.6) is isomorphic to an infinite dimensional Lie algebra. The corresponding Lie group provides a set of nonlinear invariant transformations of Eq. (7.2.2), which enable building solutions depending continuously on a denumerable set of parameters from a given solution.

Consider the perturbations $\delta' H_s$ and $\delta H'_s$ at $\delta H = h_n^\Gamma$. By induction we can prove that

$$\begin{aligned} \delta H'_{s+1} &= h_n^\Gamma H'_s - h_{n+1}^\Gamma H'_{s-1} + \ldots + (-1)^n h_{n+s}^\Gamma, \\ \delta' H_{s+1} &= {}'H_s h_n^\Gamma + {}'H_{s-1} h_{n+1}^\Gamma + \ldots + h_{n+s}^\Gamma. \end{aligned} \tag{7.2.8}$$

Bearing these formulas in mind, we consider the commutator of two infinitesimal transformations:

$$\begin{aligned} &\delta_n^{\Gamma_2}(\delta_m^{\Gamma_1} H) - \delta_m^{\Gamma_1}(\delta_n^{\Gamma_2} H) \\ &= \Gamma_1 h_{m+n-1}^{\Gamma_2} + h_1^{\Gamma_1} h_{m+n-2} + \ldots + h_{m+n-2}^{\Gamma_1} h_1^{\Gamma_2} \\ &+ h_{m+n-1}^{\Gamma_1}\Gamma_2 - \Gamma_2 h_{m+n-1}^{\Gamma_1} - h_1^{\Gamma_2} h_{m+n-2}^{\Gamma_1} \\ &- \ldots - h_{m+n-2}^{\Gamma_2} h_1^{\Gamma_1} - h_{m+n-1}^{\Gamma_2}\Gamma_1. \end{aligned} \tag{7.2.8$'$}$$

By induction we can show the validity of

$$\Gamma_1 h_m^{\Gamma_2} + h_1^{\Gamma_1} h_{m-1}^{\Gamma_2} + \ldots + h_{m-1}^{\Gamma_1} h_1^{\Gamma_2} + h_m^{\Gamma_2}\Gamma_1 = h_m^{\Gamma_1\Gamma_2}. \tag{7.2.9}$$

Hence, combining (7.2.8′) with (7.2.9) we have

$$\delta_n^{\Gamma_2}(\delta_m^{\Gamma_1} H) - \delta_m^{\Gamma_1}(\delta_n^{\Gamma_2} H) = h_{m+n-1}^{\Gamma_1\Gamma_2 - \Gamma_2\Gamma_1}.$$

Thus, the infinitesimal transformations (7.2.6) do form an algebra, which can be built by successive application of transformations h_2^Γ, provided that matrices Γ belong to a matrix algebra.

Let us now turn to transformations that reflect the fact that the matrices ${}'H_n$ and H'_n $(n \geqslant 2)$ are determined to within constant matrices.

By induction we can show that the matrices ${}'H_s$ and H'_s $(s < n)$ do not change under the transformation ${}'H_n \to {}'H_n + \Gamma$, $H'_n \to H'_n +$

Γ, with

$$\delta' H_{n+1} = \Gamma H, \ \ldots, \ \delta' H_{n+k} = \Gamma' H_k,$$
$$\delta H'_{n+1} = H\Gamma, \ \ldots, \ \delta H'_{n+k} = H'_k \Gamma.$$

We label the respective perturbations by $-m$:

$$\delta^{\Gamma}_{-m}{}' H_{m+k} = \Gamma' H_k, \quad \delta^{\Gamma}_{-m} H'_{m+k} = H'_k \Gamma.$$

Consider now perturbations of this type at $m > n$ for (7.2.6). We note that $\delta^{\Gamma_1}_{-n} h^{\Gamma}_{m} = h^{\Gamma\Gamma_1}_{m-n}$.

For the neutrino electrovacuum ($N = 3$) it follows from the definition of $H_A{}^B$ (A, $B = 1, 2$) that these components of matrix $H_a{}^b$ (a, $b = 1, 2, 3$) are determined to within arbitrary constant terms:

$$H_A{}^B \rightarrow H_A{}^B + w_0 \delta_A{}^B + i\overset{\circ}{\Psi}_A{}^B;$$

here w_0 and $\overset{\circ}{\psi}_A{}^B$ are arbitrary real constants. Similarly, the components Φ_A are determined to within arbitrary complex terms[3]

$$\Phi_A \rightarrow \Phi_A + \alpha^*_A, \quad H_{AB} \rightarrow H_{AB} - 2\alpha_A \Phi_B - \alpha_A \alpha^*_B.$$

We will now exponentiate the infinite dimensional algebra. We multiply Eqs. (7.2.3) by $(-is)^n$, with s a complex parameter, and sum over n from 0 to ∞, assuming all along that $H'_0 = 1$ and $H'_{-n} = 0$ $(n \geqslant 1)$.

We introduce in the ordinary way the generating matrix potential

$$F'(s, r, q) = \sum_{n=0}^{\infty} (-is)^n H'_n(r, q).$$

Then from (7.2.3) it follows that F' satisfies the equation

$$\nabla F' = -is(\nabla H) F' + 2(q \nabla F' - r \tilde{\nabla} F').$$

The equation can be solved for $\nabla F'$, which yields

$$[(1 - 2sq)^2 + (2rs)^2] \nabla F' = is [2rs \tilde{\nabla} H - (1 - 2qs) \nabla H]. \tag{7.2.10}$$

From Eq. (7.2.3′) it follows that F' satisfies the equation $2ir \nabla^2 F' = \nabla H \tilde{\nabla} F'$. In Eq. (7.2.10) it is expedient to go over to complex variables $z = r + iq$ and $z^* = r - iq$. Then from (7.2.10) we get

$$\frac{\partial F'}{\partial z} = -\frac{is}{1+2isz} \frac{\partial H}{\partial z} F', \quad \frac{\partial F'}{\partial z^*} = -\frac{is}{1-2isz^*} \frac{\partial H}{\partial z^*} F'. \tag{7.2.10′}$$

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[3] In other words, we can make the following transformation in the hierarchy of potentials: $H'_n \rightarrow H'_n + H'_{n-1} C_1 + \ldots + C_n$, $n = 1, 2, \ldots$, where C_i are constant matrices, with $\varepsilon C_1 - C^{\dagger}_1 \varepsilon = 0$.

We now introduce the left generating matrix potential

$$'F(s,\ r,\ q)=\sum_{m=0}^{\infty}(is)^{m}\,'H_{m}(r,\ q).$$

From the recurrence formulas (7.2.4) it follows that the matrix $'F(s,\ r,\ q)$ satisfies the equation

$$\nabla'F = is'F\,\nabla H + 2\,(q\,\nabla'F - r\,\tilde{\nabla}'F),$$

whence

$$[(1-2sq)^{2}+(2rs)^{2}]\,\nabla'F = -is'F\,[2rs\,\tilde{\nabla}H-(1-2qs)\,\nabla H].$$

From (7.2.4) is follows that $'F$ satisfies the equation

$$2ir\,\nabla^{2}\,'F=\nabla'F\,\tilde{\nabla}H.$$

We now calculate the product $'FF'$;

$$'FF'=\sum_{n=0}^{\infty}(is)^{n}\,['H_{n}-'H_{n-1}H+'H_{n-2}H'_{2}+\ldots+(-1)^{n}H'_{n}].$$

According to (7.2.5), the coefficients of $(is)^{n}$ at $n\geqslant 1$ are zero and, hence,

$$'F=(F')^{-1}. \qquad (7.2.11)$$

Therefore, from now on we will drop the prime on F', i.e. $F'\equiv F$.

Equation (7.2.10) has an important corollary. We multiply both sides of Eq. (7.2.10) into F^{-1} from the right and take the trace of the resulting matrix equation. This yields

$$[(1-2sq)^{2}+4s^{2}r^{2}]\,\nabla\ln(\det F)$$

$$=is\,[2rs\,\tilde{\nabla}\,H_{a}{}^{a}+(1-2qs)\,\nabla H_{a}{}^{a}]. \qquad (7.2.12)$$

Now we employ the fact that, according to (7.1.22), the trace $H_{a}{}^{a}$ is equal to $2\,(w+iq)$. When we go over to complex variables, from (7.2.12) we have

$$(1+2isz)\,\partial\,[\ln(\det F)]/\partial z=-2is\,(\partial w/\partial z+1/2), \qquad (7.2.13)$$

$$(1-2isz^{*})\,\partial\,[\ln(\det F)]/\partial z^{*}=-2is\,(\partial w/\partial z^{*}-1/2).$$

Integrating (7.2.13), we finally have

$$\det F=e^{-i\sigma}/\lambda;\quad \lambda=\sqrt{(1-2sq)^{2}+(2sr)^{2}},$$

$$\sigma=2s\left[\int\frac{\partial w}{\partial z}\,\frac{dz}{1+2izs}+\int\frac{\partial w}{\partial z^{*}}\,\frac{dz^{*}}{1-2iz^{*}s}\right]. \qquad (7.2.13')$$

Let us now consider the infinitesimal transformations $\delta H'_{s+1}$, $\delta' H_{s+1}$ generated by the infinite dimensional group we are studying. We introduce the notation $H_{s,t}='H_{s}H'_{t}\,(-1)^{t}$. Then, combining (7.2.8) with (7.2.6), we have

$$\delta H_{s+1,\,t+1} = -\sum_{k=0}^{n-1} H_{s+1,\,k}\Gamma_{n-1}\sum_{i=0}^{t} H_{t+n-i-k,\,i}$$

$$-H_{s+1,\,n+t}\Gamma_{n-1} + \sum_{k=0}^{n-1}\sum_{i=0}^{s} H_{i,\,n+s-i-k}\Gamma_{n-1}H_{k,\,t+1}$$

$$+\Gamma_{n-1}H_{s+n,\,t+1}. \qquad (7.2.14)$$

Here Γ_{n-1} is the matrix that determines h_n^Γ (see definition (7.2.6)).

It is interesting that Eq. (7.2.14) can be simplified if we introduce the matrices T_{mn} in the following way:

$$\begin{aligned} &-T_{m,n} + T_{m-1,\,n+1} = H_{m,n} \text{ at } m \geqslant 1,\ n \geqslant 1, \\ &T_{0,n} = -H_{0,n} = -(-1)^n H'_n \text{ at } n \geqslant 1, \qquad (7.2.15) \\ &T_{m-1,1} = {'H_m},\ m \geqslant 1,\ T_{m,n} = 0,\ m < 0,\ n < 1. \end{aligned}$$

To demonstrate that the transformation from $H_{m,n}$ to $T_{m,n}$ is non-singular, we multiply (7.2.15) into $(is)^m\,(it)^n$ and sum over m and n from 0 to ∞. Next we introduce the generating matrix functions

$$T(s,\ t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n}\,(is)^m\,(it)^n \quad \text{and}$$

$$H(s,\ t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} H_{m,n}\,(is)^m\,(it)^n.$$

According to the definition of $H_{m,n}$,

$$H(s,\ t) = {'F}(s)\,F'(t) = F^{-1}(s)\,F(t).$$

From (7.2.15) we have

$$T(s,\ t)\,(-1 + s/t) = F^{-1}(s)\,F(t) - 1,$$

whence

$$T(s,\ t) = t/(s-t)\,[F^{-1}(s)\,F(t) - 1]. \qquad (7.2.15')$$

Taking the coefficients of the like terms in powers of s and t, we get a unique formula for $T_{m,n}$ in terms of $H_{m,n}$.

If we substitute (7.2.15) into Eqs. (7.2.14) at $m \geqslant 1$, we can easily find that

$$\delta T_{l,\,m} = \Gamma_{n-1}T_{n-1+l,\,m} - T_{l,\,n-1+m}\Gamma_{n-1}$$

$$-\sum_{k=1}^{n-1} T_{l,\,k}\Gamma_{n-1}T_{n-1-k,\,m}. \qquad (7.2.16)$$

In an infinitesimal transformation, in general, all the Γ_n are nonzero; hence, from (7.2.16) at $m \geqslant 1$ we have

$$\delta T_{l,m} = \sum_{k=l}^{\infty} \Gamma_{-l+k} T_{k,m} - \sum_{k=m}^{\infty} T_{l,k} \Gamma_{k-m}$$

$$- \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} T_{l,k} \Gamma_{k+n} T_{n,m}. \qquad (7.2.16')$$

This expression is in matrix form and has the same dimension N as the matrix H. We now form infinite dimensional matrices Ω and λ, whose elements are matrices of dimension N:

$\Omega_{i,k}$ $(-\infty < i < +\infty,\ -\infty < k < +\infty)$
$\Omega_{m,n} = \delta_{m,n}$ at $m \geqslant 0,\ n \geqslant 0,\ \Omega_{m,-n} = 0,\ n \geqslant 1,\ -\infty < m < \infty,$
$\Omega_{-m,n} = T_{m,n}$ at $m \geqslant 0,\ n \geqslant 1,\ \Omega_{-m,0} = 0,\ m \geqslant 1;$
λ_{ik} $(-\infty < i < +\infty,\ -\infty < k < +\infty),$
$\lambda_{ik} = \Gamma_{i-k}$ at $-\infty < k \leqslant i < \infty,$
$\lambda_{ik} = 0$ at $-\infty < i < k < \infty.$

Then we are able to write Eqs. (7.2.16′) in the form of one matrix equation:

$$\delta\Omega = \lambda\Omega - \Omega\lambda\Omega. \qquad (7.2.17)$$

According to the definition, matrix Ω has the form

$$\Omega = \left(\begin{array}{c|c} 0 & T \\ \hline 0 & E \end{array}\right)$$

with E the unit matrix. While λ is a triangular matrix:

$$\lambda = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \Gamma_0 & 0 & 0 & 0 & \cdots \\ \cdots & \Gamma_1 & \Gamma_0 & 0 & 0 & \cdots \\ \cdots & \Gamma_2 & \Gamma_1 & \Gamma_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It can easily be shown that the matrix equation (7.2.17) has the solution

$$\Omega\,(\overset{\circ}{Q} - Q + E) = Q, \qquad (7.2.18)$$

where $\overset{\circ}{Q}$ is an arbitrary constant matrix, E is the unit matrix, and Q is the solution of the equation

$$\delta Q = \lambda Q, \text{ or } \delta Q_{n,k} = \sum_{s=-\infty}^{i} \Gamma_{n-s} Q_{s,k}. \qquad (7.2.19)$$

To find the solution of (7.2.19) at a fixed k, we multiply both sides of Eq. (7.2.19) into $(it)^n$, sum over n from $-\infty$ to $+\infty$, and put $Q_k(t) = \sum_{n=-\infty}^{+\infty} Q_{n,k}(it)^n$, $\Gamma(t) = \sum_{k=0}^{\infty} \Gamma_k (it)^{-k}$. Then we can write (7.2.19) in the form $\delta Q_k = \Gamma(t) Q_k$, where Q_k and $\Gamma(t)$ are matrices of dimension N, from which we can easily find that

$$Q_k = (\exp \gamma(t)) \mathring{Q}_k, \quad \Gamma(t) = \delta\gamma(t). \tag{7.2.20}$$

It is expedient to identify matrix $\mathring{Q}$ with $\mathring{\Omega}$, which corresponds to the unit element of the group. Matrix Ω is obtained from $\mathring{\Omega}$ via a nonlinear group transformation. Let us introduce the notation $\tau(t) \equiv \exp\gamma(t) - 1$. Then (7.2.20) can be written as

$$-\mathring{\Omega}_{l,m} + Q_{l,m} = \sum_{s=-\infty}^{l} \tau_{l-s} \mathring{\Omega}_{s,m}, \quad \tau(t) = \sum_{k=0}^{\infty} \tau_k (it)^{-k}. \tag{7.2.21}$$

Allowing for (7.2.21), we can write (7.2.18) at $n \geqslant 1$ as

$$T_{m,n} - \mathring{T}_{m,n} = \sum_{s=m}^{\infty} \tau_{s-m} \mathring{T}_{s,n} - \sum_{s=n}^{\infty} T_{m,s}\tau_{s-n} - \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} T_{m,k}\tau_{k+l}\mathring{T}_{l,n}. \tag{7.2.22}$$

Now we put $m=0$, multiply both sides of (7.2.22) into $(it)^n$, sum over n from 1 to ∞, and get

$$-F(t) + \mathring{F}(t) + \sum_{n,l,m}^{\infty} T_{0,n}\tau_{n+m}\mathring{T}_{m,l}(it)^l$$
$$+ \sum_{l=1}^{\infty}\sum_{s=l}^{\infty} T_{0,s}\tau_{s-l}(it)^l = \sum_{l=1}^{\infty}\sum_{s=0}^{\infty} \tau_s \mathring{T}_{s,l}(it)^l. \tag{7.2.23}$$

Consider a closed curve L that separates a simply connected domain D_+ (the origin of coordinates belongs to D_+) from a simply connected domain D_- (the point at infinity belongs to D_-). We assume that in the domain $D_+ + L$ the functions $F(t)$ and $\mathring{F}(t)$ are analytic, while in the domain $D_- + L$ the function $\tau(t)$ is analytic. From (7.2.13′) it follows that $t = i/2z$ and $t = -i/2z^*$ are singular points of $F(t)$, and for this reason both must lie in D_-.

We now employ the fact that $2\pi i \tau_n = \int_L (is)^n \tau(s)\, ds/s$ and obtain

$$\sum_{l=1}^{\infty}\sum_{n=l}^{\infty} T_{0,n}\tau_{n-l}(it)^l = \sum_{n=1}^{\infty} \int T_{0,n}(is)^n \sum_{l=1}^{n} (t/s)^l \tau(s)\, ds$$
$$= \int_L \{[F(s) - \mathring{F}(t)]/[s(s-t)]\} \tau(s)\, t\, ds. \tag{7.2.24}$$

Using formula (7.2.15′) and combining (7.2.23) with (7.2.24),

$$F(t)\left(1-\frac{1}{2\pi i}\int\limits_L \tau(s)\,\frac{t\,ds}{s(s-t)}\right)$$

$$+\frac{1}{2\pi i}\int\limits_L F(s)\,\tau(s)\,\overset{\circ}{F}{}^{-1}(s)\,\overset{\circ}{F}(t)\,\frac{t\,ds}{s(s-t)}=0. \qquad (7.2.25)$$

If we use the fact that $\tau(s)$ is analytic in D_- and $F(s)\,\overset{\circ}{F}{}^{-1}(s)$ is analytic in D_+, we have

$$\frac{1}{2\pi i}\int\limits_L \tau(s)\,\frac{t\,ds}{s(s-t)}=\begin{cases}-\tau(t) & \text{at } t\in D_-,\\ 0 & \text{at } t\in D_+;\end{cases}$$

$$\frac{1}{2\pi i}\int\limits_L F(s)\,\overset{\circ}{F}{}^{-1}(s)\,\frac{t\,ds}{s(s-t)}=\begin{cases}-1 & \text{at } t\in D_-,\\ -1+F(t)\,\overset{\circ}{F}{}^{-1}(t) & \text{at } t\in D_+.\end{cases}$$

We introduce the notation $\chi_-\equiv F(t)\,(1+\tau(t))\,\overset{\circ}{F}{}^{-1}(t)$. From (7.2.25) it follows that

$$\begin{aligned}&\int\limits_L \chi_-(s)\,\frac{t\,ds}{s(s-t)}=0, \quad t\in D_+,\\ &\chi_-(t)+\frac{1}{2\pi i}\int\limits_L \chi_-(s)\,\frac{t\,ds}{s(s-t)}=0, \quad t\in D_-.\end{aligned} \qquad (7.2.25')$$

The above formulas show that the function $\chi_-(s)$ given on L admits of an analytic continuation into D_-. The respective function in D_- we denote by $\chi_-(t)$.[4]

If we now introduce the notations $\chi_+(s)=F(s)\,\overset{\circ}{F}{}^{-1}(s)$ and $G(s)\equiv\overset{\circ}{F}(s)\,(1+\tau(s))\,\overset{\circ}{F}{}^{-1}(s)$, then we can interpret (7.2.25′) in the following manner: we wish to find functions $\chi_+(t)$ and $\chi_-(t)$ analytic in D_+ and D_-, respectively, such that

$$\chi_-(s)=\chi_+(s)\,G(s) \qquad (7.2.26)$$

on L and

$$\chi_+(0)=1. \qquad (7.2.27)$$

This is the classical Riemann boundary problem in the theory of analytic functions. Note that $\det\chi_+=1$ in D_+ and therefore the problem is regular.

---

[4] Equations (7.2.25′) could have been derived directly from Eqs. (7.2.8) without introducing the generating matrix functions $H(s,t)$ and $T(s,t)$. Indeed, using the generating function $F(s)$, we find that Eqs. (7.2.8) yield

$$\int\limits_L [\delta F(s)\,F^{-1}(s)+F(s)\,\Gamma(s)\,F^{-1}(s)]\,ds/[s(s-t)]=0 \quad t\in D_+,$$

which is the variation of matrix equations (7.2.25′).

If we use the solution of Riemann's problem for a fixed L, we can obtain the solution to the initial equation (7.2.2) not for all values of r and q but only in the domain $i/2r \in D_-$, i.e. in a restricted domain that includes the origin of coordinates.

We can easily prove that when a solution exists, it must be unique. Indeed, suppose that χ'_-, χ'_+ is another solution. Then on L we have $\chi_+^{-1}\chi_- = (\chi'_+)^{-1}\chi'_- = G(s)$; hence $\chi_-(\chi'_-)^{-1} = \chi_+(\chi'_+)^{-1}$.

The function that is defined as $\chi_-(\chi'_-)^{-1}$ in D_- and as $\chi_+(\chi'_+)^{-1}$ in D_+ is analytic in the entire complex plane and, therefore, by Liouville's theorem is a constant. This constant is equal to the unit matrix, i.e. the matrix $\chi_+(\chi'_+)^{-1}$ at $t = 0$ is the unit matrix by definition. Whence $\chi_+ = \chi'_+$ in D_+ and $\chi_- = \chi'_-$ in D_-.

We note that in the absence of neutrino fields, the Riemann problem and the integral equations (7.2.25′) were first formulated by Hauser and Ernst [4], who based their reasoning on the matrix potentials of Kinnersley and Chitre [3, 5]. Our method of introducing matrix potentials does not contain the operation of complex conjugation and is based solely on the set of equations (7.2.2).[5]

In the absence of neutrino fields, we can use the Minkowski space-time as the bare solution. For this case we can easily build the matrix $\overset{\circ}{F}(t)$. From (7.1.18) it follows that at $w \neq 0$ the flat space-time is not a solution. A generalization of the flat solution is

$$(g_{AB}) = \begin{pmatrix} 1 & w \\ w & w^2 - r^2 \end{pmatrix}. \qquad (7.2.28)$$

For the complex matrix potential according to (7.1.19), we have

$$H_{11} = 1,\ H_{12} = 2(w + iq),\ H_{21} = 0, \qquad (7.2.29)$$

$$H_{22} = w^2 - r^2 + 2iqw + 2\int z^* (\partial w/\partial z^*)\, dz^* - 2\int z\, (\partial w/\partial z)\, dz.$$

To find the components of the generating matrix function $\overset{\circ}{F}(t)$ corresponding to solution (7.2.28) we use Eq. (7.2.10′):

$$\frac{\partial \ln F_{1B}}{\partial z} = -\frac{2it}{1+2izt}\left(\frac{\partial w}{\partial z} + \frac{1}{2}\right),$$

$$\frac{\partial \ln F_{1B}}{\partial z^*} = -\frac{2it}{1+2izt}\left(\frac{\partial w}{\partial z^*} - \frac{1}{2}\right), \qquad (7.2.30)$$

$$\partial F_{2B}/\partial z = (\partial F_{1B}/\partial z)(w - r),\ \partial F_{2B}/\partial z^* = (\partial F_{1B}/\partial z^*)(w + r).$$

Integrating (7.2.30), we obtain

$$F_{1B} = (c_B(t)/\lambda)\, e^{-i\sigma},\ F_{2B} = (c_B(t)/\lambda)\, e^{-i\sigma}(w + iq + 1/2it) + d_B(t)$$

---

[5] Equations (7.2.2) are differential corollaries of the initial equations (7.1.17) and (7.1.20); therefore, we have to adjust the constructed algebra to those equations. This leads to additional restrictions on $\tau(s)$, which for $n = 3$ must be $1 + \tau(s) = \exp[\gamma(s)(\varepsilon - is\,\Pi/2)]$ (see the footnote on page 194), with $\gamma(s) = \gamma^+(s)$.

Here c_B and d_B are arbitrary functions of parameter t, and the expressions for λ and σ are given by formula (7.2.13′). By the definition of F_{AB}, the following is true:

$$F_{AB}(0) = \varepsilon_{AB}, \quad \partial F_{AB}/\partial t\,|_{t=0} = -iH_{AB}. \tag{7.2.31}$$

The matrix functions H'_n ($n = 2, 3, \ldots$) are determined to within constant complex matrices. For this reason we can take the simplest expressions for $c_B(t)$ and $d_B(t)$ for which conditions (7.2.31) are met: $c_1 = -it$, $c_2 = 1$, $d_1 = -1/2$, $d_2 = -1/2it$.

Hence the matrix $F_A{}^B(t)$ for solution (7.2.28) has the form

$$\lambda \exp(i\sigma)\, F_A{}^B(t) = \begin{pmatrix} 1 & it \\ w+iq+(1-\lambda\exp(i\sigma))/2it & it\,(w+iq)+(1+\lambda\exp(i\sigma))/2 \end{pmatrix} \tag{7.2.32}$$

7.3 EXACT SOLUTIONS OF NEUTRINO VACUUM EQUATIONS

Few exact solutions of the simultaneous set of Einstein-Weyl equations are known. Golubyatnikov [6] found an exact solution with a group of motions G_3 on V_2. Solutions with an isotropic Killing vector (the case of pure radiation) and a zero energy-momentum tensor are also known (see [7, 8]). Griffiths [9] found solutions that describe the collision of a stepwise neutrino wave and a delta-like gravitational wave. Repchenkov [10] found a solution that depends on one variable only and which, from considerations of limiting transitions with respect to parameters on which the solution depends, was interpreted by him as collision of neutrino waves.

Below we present a class of new exact solutions found by the author. In Sec. 7.3.1 we reduce the problem to an analysis of a first-order equation; in Sec. 7.3.2 we study a self-similar collision of neutrino waves; in Sec. 7.3.3 we give a steady-state solution with local sources of the neutrino field; finally, in Sec. 7.3.4 we describe ways of generating exact solutions, i.e. methods that lead to solutions with any number of arbitrary constants, which are based on the theory we have developed in Sec. 7.2.

7.3.1. A Class of Solutions Described by First-Order Equations. Let us now consider the class of exact solutions of the equation to which, as we have shown earlier (see (7.1.23)), we can reduce the Einstein-Weyl set of equations in the stationary axisymmetric case:

$$r\,\nabla^2\mathscr{E} + \nabla\mathscr{E}\,(\nabla r + i\,\nabla\tilde{w}) - 2r\,(\nabla\mathscr{E})^2/(\mathscr{E} + \mathscr{E}^*) = 0, \tag{7.3.1}$$

where $\tilde{w}$ is a real-valued harmonic function, and $\nabla \equiv (\partial/\partial r,\ \partial/\partial q)$. We write Eq. (7.3.1), introducing new coordinates $z = r + iq$ and $z^* = r - iq$:

$$\frac{\partial^2 \mathscr{E}}{\partial z\, \partial z^*} + \frac{1}{2(z+z^*)}\left(\frac{\partial \mathscr{E}}{\partial z} + \frac{\partial \mathscr{E}}{\partial z^*}\right) - \frac{2}{\mathscr{E}+\mathscr{E}^*}\frac{\partial \mathscr{E}}{\partial z}\frac{\partial \mathscr{E}}{\partial z^*}$$

$$+ \frac{i}{2(z+z^*)}\left[\frac{\partial \mathscr{E}}{\partial z}\frac{\partial \tilde{w}}{\partial z^*} + \frac{\partial \mathscr{E}}{\partial z^*}\frac{\partial \tilde{w}}{\partial z}\right]$$

$$= \frac{\partial^2 \mathscr{E}}{\partial z\, \partial z^*} + \left[\frac{\partial \mathscr{E}}{\partial z}\left(\frac{1}{2} + i\frac{\partial \tilde{w}}{\partial z^*}\right) + \frac{\partial \mathscr{E}}{\partial z^*}\left(\frac{1}{2} + i\frac{\partial \tilde{w}}{\partial z}\right)\right] \Big/ (z+z^*)$$

$$- \frac{2}{\mathscr{E}+\mathscr{E}^*}\frac{\partial \mathscr{E}}{\partial z}\frac{\partial \mathscr{E}}{\partial z^*}. \tag{7.3.2}$$

Let us show that the solution to the equation

$$\frac{\partial \mathscr{E}}{\partial z} - \frac{\mathscr{E}+\mathscr{E}^*}{(z+z^*)}\left(\frac{1}{2} + i\frac{\partial \tilde{w}}{\partial z}\right) = 0 \tag{7.3.3}$$

satisfies Eq. (7.3.1) provided that $\partial \mathscr{E}^*/\partial z^* = (\partial \mathscr{E}/\partial z)^*$. Indeed, differentiation of (7.3.3) with respect to z^* yields

$$\frac{\partial^2 \mathscr{E}}{\partial z\, \partial z^*} + \frac{1}{(z+z^*)}\left[\frac{\partial \mathscr{E}}{\partial z} - \frac{1}{2}\frac{\partial \mathscr{E}}{\partial z^*}\left(\frac{1}{2} + i\frac{\partial \tilde{w}}{\partial z}\right)\right.$$

$$\left. - \frac{\partial \mathscr{E}^*}{\partial z^*}\left(\frac{1}{2} + i\frac{\partial \tilde{w}}{\partial z}\right)\right] = 0. \tag{7.3.4}$$

The required assertion follows if we compare (7.3.2) and (7.3.4), and from the relationship

$$\partial \mathscr{E}^*/\partial z^*\ (1 + i\, \partial \tilde{w}/\partial z) = (\partial \mathscr{E}/\partial z)\ (1 - i\, \partial \tilde{w}/\partial z^*),$$

which is valid on the basis of the above condition, Eq. (7.3.3), and the fact that $\tilde{w}$ is a harmonic function.

This class of solutions of neutrino vacuum equations is more general than that found by Bitsadze [11] for the case of a gravitational vacuum.

7.3.2. Self-Similar Collision of Neutrino Waves.[6] When there are no electromagnetic waves, the set of equations (7.1.37) after the substitutions

$$g_{11} = r \cosh W \exp V,\ g_{12} = -r \sinh W,\ g_{22} = r \cosh W \exp(-V)$$

---

[6] The results of this subsection belong to L.Yu. Blazhenova-Mikulich and the author.

can be reduced to

$$\begin{aligned}&2rW_{,uv} + r_{,u}W_{,v} + r_{,v}W_{,u} - rV_{,u}V_{,v}\sinh 2W\\&+\cosh W\,(c_1(u)\,V_{,v} - c_2(v)\,V_{,u}) = 0, \qquad (7.3.5)\\&(2rV_{,uv} + r_{,u}V_{,v} + r_{,v}V_{,u})\cosh W + 2r\,(V_{,v}W_{,u} + V_{,u}\,W_{,v})\\&+ c_2(v)\,W_{,u} - c_1(u)\,W_{,v} = 0 \quad (c_1(u) = C_{,u},\ C_2(v) = c_{,v}).\end{aligned}$$

The self-similar solutions to this set of equations are sought in the form

$$\begin{aligned}&W = W(\theta), \quad V = \alpha \ln r + \tilde{V}(\theta),\\&\theta = \arccos\,[(f(u) - g(v)/(f(u) + g(v))], \quad r = f(u) + g(v). \qquad (7.3.6)\end{aligned}$$

These exist if

$$c_1(u) = Af'(u), \quad c_2(v) = Bg'(v).$$

Then (7.3.5) is reduced to

$$\left.\begin{aligned}&\tilde{V}''\cosh W + 2\tilde{V}'W'\sinh W - 2\alpha\cot\theta W'\sinh W\\&- W'(A - B\cos\theta)/\sin\theta = 0,\\&W'' + (1/2)(\alpha^2 + 2\alpha\cot\theta\tilde{V}' - \tilde{V}'^2)\sinh 2W\\&+ \cosh W\,\{\alpha B + [(B\cos\theta - A)/\sin\theta]\,\tilde{V}'\} = 0,\end{aligned}\right\} \qquad (7.3.7)$$

where the prime means differentiation with respect to θ. As in the case of self-similar collision of electromagnetic and gravitational waves studied by Blazhennova-Mikulich and Sibgatullin [12], there is a first integral for (7.3.7):

$$(\tilde{V}')^2\cosh^2 W + (W')^2 + (\alpha\sinh W + B)^2 = c_0^2,$$

which at $\alpha = 0$ has the same form as in the case of collision of purely gravitational waves. We assume that $V'\cosh W = c_0\cos\chi$ and calculate the Ricci rotation coefficients:

$$\begin{aligned}&\lambda = -\tan(\theta/2)\exp(i\chi)\,c_0 f'(u)/2r,\\&\sigma = -\cot(\theta/2)\exp(-i\chi)\,c_0 g'(v)/2r.\end{aligned}$$

We see that the colliding waves undergo a shift in which the rotation angles of the major axes are opposite and the magnitude of the shift obeys the "conservation law" $|\lambda\sigma| = c_0^2 f'(u)\,g'(v)/4r^2$.

The obtained first integral enables reducing the order of the set of equations. For $\alpha = 0$ we introduce a new variable: $\mu = \arccos(\tanh W)$. Then (7.3.7) is reduced to

$$\begin{aligned}&\mu' = -c_0\sin\chi\sin\mu, \qquad (7.3.8)\\&\chi' = c_0\cos\mu\cos\chi + (A - B\cos\theta)/\sin\theta.\end{aligned}$$

This set of equations has singularities at $\theta = 0$ and $\theta = \pi$, i.e. the metric is singular on the surfaces $u = u_0$, $v = v_0$ where $f(u)$ and $g(v)$ vanish, respectively.

The asymptotic behavior of μ and χ at the singular points is:

$$\theta \to 0: \ \mu \approx \mu_0, \ \chi \approx \chi_0 + (A - B) \ln \tan (\theta/2);$$

$$\theta \to \pi: \ \mu \approx \text{const}, \ \chi \approx \chi_0 + (A + B) \ln \tan (\theta/2).$$

Hence, as the colliding waves approach the singular points, they undergo "shearing" vibrations with an infinitely increasing frequency.

The quantities $\chi + \mu$ and $\chi - \mu$ satisfy the equation

$$\zeta' = c_0 \cos \zeta + (A - B \cos \theta)/\sin \theta.$$

The phase of the neutrino waves can be found from the Weyl equations (7.1.2):

$$4 \arg \Phi = \chi + (2B - A) \ln |\sin \theta| - B \ln |\tan (\theta/2)| + \text{const},$$

$$4 \arg \Psi = -\chi - (2B + A) \ln |\sin \theta| - B \ln |\tan (\theta/2)| + \text{const}.$$

Thus, in the self-similar problem of colliding neutrino waves, the approach to a singularity has an oscillating behavior that is due to shearing vibrations with infinitely increasing frequency. In the problem of self-similar collision of electromagnetic waves (see [12]), notwithstanding the formal similarity of the solution to that for neutrino wave collision, the physics of the vibrations is entirely different: the vibrations appear due to transmutation, so to say, between gravitational and electromagnetic waves in the approach to a singularity (this effect was studied by Sibgatullin [13] in the approximation of nonlinear geometrical optics).

7.3.3. A Stationary Axisymmetric Solution. In the coordinates λ, μ: $q = \lambda\mu$, and $r = [(\lambda^2 - 1)(1 - \mu^2)]^{1/2}$, Eq. (7.3.1) becomes

$$\frac{\partial}{\partial\lambda}\left[(\lambda^2 - 1)\frac{\partial\mathscr{E}}{\partial\lambda}\right] - \frac{2(\lambda^2-1)}{(\mathscr{E}+\mathscr{E}^*)}\left(\frac{\partial\mathscr{E}}{\partial\lambda}\right)^2 + i\sqrt{\frac{\lambda^2-1}{1-\mu^2}}\,\frac{\partial\tilde{w}}{\partial\lambda}\frac{\partial\mathscr{E}}{\partial\lambda}$$
$$+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\mathscr{E}}{\partial\mu}\right] - \frac{2(1-\mu^2)}{(\mathscr{E}+\mathscr{E}^*)}\left(\frac{\partial\mathscr{E}}{\partial\mu}\right)^2 + i\sqrt{\frac{1-\mu^2}{\lambda^2-1}}\,\frac{\partial\tilde{w}}{\partial\mu}\frac{\partial\mathscr{E}}{\partial\mu} = 0. \tag{7.3.9}$$

In the λ, μ-coordinates the Laplace equation is

$$\sqrt{\lambda^2-1}\,\partial\left(\sqrt{\lambda^2-1}\,\frac{\partial\tilde{w}}{\partial\lambda}\right)\Big/\partial\lambda + \sqrt{1-\mu^2}\,\partial\left(\sqrt{1-\mu^2}\,\frac{\partial\tilde{w}}{\partial\mu}\right)\Big/\partial\mu = 0.$$

The simplest solution for this equation that tends to zero as $r \to \infty$ (this is necessary for the space-time to be asymptotically flat) is

$$\tilde{w} = \gamma\sqrt{1-\mu^2}\,(\lambda - \sqrt{\lambda^2-1}).$$

Let us look for the solution to Eq. (7.3.9) in the form $\mathcal{E} = \mathcal{E}(\lambda)$. We put $\lambda = \cosh\sigma$ and obtain

$$\frac{d^2\mathcal{E}}{d\sigma^2} + \coth\sigma \frac{d\mathcal{E}}{d\sigma} - \frac{2}{\mathcal{E}+\mathcal{E}^*}\left(\frac{d\mathcal{E}}{d\sigma}\right)^2 + i\gamma \frac{d\mathcal{E}}{d\sigma}\frac{\exp(-\sigma)}{\sinh\sigma} = 0,$$

or, separating the real and imaginary parts,

$$f'' + \coth\sigma f' - \frac{(f'^2 - \psi'^2)}{f} - \gamma\psi' e^{-\sigma}/\sinh\sigma = 0,$$

$$\psi'' + \coth\sigma\psi' - 2f'\psi'/f + \gamma f' e^{-\sigma}/\sinh\sigma = 0$$

(the prime stands for differentiation with respect to σ). We multiply the first equation by f', the second by ψ', and add the two. This yields

$$\sinh^2\sigma\,(f'^2 + \psi'^2) = c_0^2 f^2.$$

Now we introduce the variable ζ:

$$c_0 \sin\zeta = (f'/f)\sinh\sigma, \quad c_0\cos\zeta = (\psi'/f)\sinh\sigma,$$

which satisfies the equation

$$\zeta' = -c_0\cos\zeta/\sinh\sigma + \gamma e^{-\sigma}/\sinh\sigma,$$

and the variable θ such that $d\theta = d\sigma/\sinh\sigma$. Then we have

$$d\zeta/d\theta = c_0\cos\zeta - \gamma(\cosh\theta - 1)/\sinh\theta.$$

It is worth noting that for an entirely different situation we have arrived at the same equation as in the case of self-similar collision of neutrino waves (to within the substitution of a hyperbolic sine for an ordinary sine).

7.3.4. Generating Solutions by a Shift along an Orbit of the Infinite Dimensional Group. The integral matrix equation (7.2.25′) for $t \in D_+$ is reduced to an integral equation for one sought function in two sufficiently general cases: (i) when matrix τ has only one nonzero, off-diagonal component, and (ii) when matrix τ is diagonal. Although an arbitrary matrix τ can be represented by a product of four matrices of type (i), nevertheless a study of transformations of type (ii) enables us to obtain physically interesting solutions by solving an integral equation only once.

In this section we will derive the integral equations and give their solution for matrix $\tau(s)$ that is an arbitrary rational function analytic at the point at infinity for cases (i) and (ii), respectively. If we choose the neutrino field w in the form $w = \pm\rho$, it proves possible to find a solution for an arbitrary shift along the orbit of the infinite dimensional group.

(i) Let us consider the integral matrix equation (7.2.25) at $t \in D_+$ for the component $F_1{}^2(t)$ on the assumption that matrix τ has only one nonzero component $\tau_2{}^1(s)$. If we put $\tau_2{}^1(s) \equiv T(s)$ and $F_1{}^2(s) \equiv$

$-Y(s)s$, we have a Fredholm integral equation of the second kind:

$$Y(t)-\overset{\circ}{Y}(t)=\frac{1}{2\pi i}\int\limits_{L} Y(s)\,T(s)\,M(s,\,t)\,ds, \tag{7.3.10}$$

with

$$M(s,\,t)\equiv\lambda(s)\exp(i\sigma(s))\,[\overset{\circ}{F}{}_2^2(s)\,\overset{\circ}{F}{}_1^2(t)-\overset{\circ}{F}{}_1^2(s)\,\overset{\circ}{F}{}_2^2(t)]\,\frac{1}{(s-t)}\,. \tag{7.3.11}$$

Here $\overset{\circ}{F}{}_1^2\,(t)$ and $\overset{\circ}{F}{}_2^2\,(t)$ are the components of the generating matrix F for the initial bare solution, from which we will generate all other solutions. From the fact that $T(s)$ is bounded as $s\to\infty$ and rational it follows that this function can be written as

$$T(s)=\alpha+\sum_{m=1}^{N}\sum_{n=0}^{\alpha m} a_n^m/(s-u_m)^{n+1}. \tag{7.3.12}$$

The complex constants a_n^m and u_m are restricted by the condition that after $T(s)$ is reduced to a common denominator, it can be written in the form of a ratio of two polynomials with real coefficients and such that the degree of the polynomial in the numerator is no greater than that of the polynomial in the denominator.

After we integrate by parts and calculate the residues at points $s=u_m$ in (7.3.10) and (7.3.12), we obtain

$$Y(t)-\overset{\circ}{Y}(t)=\sum_{m=1}^{N}\sum_{n=0}^{\alpha m}\frac{a_n^m}{n!}\,\frac{\partial^n}{\partial s^n}\,(Y(s)\,M(s,\,t)_{s=u_m}. \tag{7.3.13}$$

Let us denote $\partial^k Y(s)/\partial s^k$ at $s=u_m$ by Y_{mk}. From (7.3.13) it follows that

$$\begin{aligned} Y(t)-\overset{\circ}{Y}(t)&=\sum_{m=1}^{N}\sum_{k=0}^{\alpha m} Y_{mk}\varphi^{mk}(t),\\ \varphi^{mk}(t)&=\sum_{n=0}^{\alpha m-k} a_{n+k}^m\,\frac{\partial^n}{\partial s^n}\,\frac{M(s,\,t)}{k!n!}\bigg|_{s=u_m}. \end{aligned} \tag{7.3.14}$$

We differentiate (7.3.14) q times with respect to t and put $t=u_l$ (q runs from zero to α_l, with $l=1,\,2,\,\ldots,\,N$). Then from (7.3.14) we obtain a closed linear set of algebraic equations to determine

Y_{mk}:

$$Y_{lq}-\overset{\circ}{Y}_{lq}=\sum_{m=1}^{N}\sum_{k=0}^{\alpha m} Y_{mk}Q_{lq}^{mk}, \quad q=0, \ldots, \alpha_l, \; l=1, \ldots, N, \tag{7.3.15}$$

$$Q_{lq}^{mk}\equiv \sum_{n=0}^{\alpha m-k} \frac{1}{n!}\frac{1}{k!} a_{n+k}^{m} \left(\frac{\partial^{n+q}}{\partial s^n \partial t^q} M(s, t)\right)_{s=u_m,\; t=u_l}.$$

The sought solution $\mathscr{E} = \mathscr{E}(r, q)$ of Eq. (7.3.1) is expressed in terms of the solution to (7.3.15) in the following manner:

$$\mathscr{E}=iY(0)=\overset{\circ}{\mathscr{E}}+i\sum_{m=1}^{N}\sum_{k=0}^{\alpha m} Y_{mk}\sum_{n=0}^{\alpha m-k}\frac{1}{k!n!}a_{n+k}^{m}\frac{\partial^n}{\partial s^n}[\overset{\circ}{Y}(s)\lambda(s)e^{i\sigma}]_{s=u_m}. \tag{7.3.16}$$

If we use (7.2.32) as the initial solution, we have

$$\mathscr{E}=1+\sum_{m=1}^{N}\sum_{k=0}^{\alpha m}(Y_{mk}/k!)\,a_k^m. \tag{7.3.16$'$}$$

In the absence of a neutrino field solution (7.3.16′) of Eqs. (7.3.15) includes as particular cases the solutions found earlier by Sato and Tomimatzu [14], Kinersley and collaborators [3, 5, 15], and Hauser and Ernst [4].

(ii) We note that in the general case the matrix $1+\tau(s)$, being the exponent of a matrix with a zero trace, has a determinant equal to unity. Therefore, if matrix $1+\tau(s)$ is diagonal, it must necessarily have the following form:

$$1+\tau(s)=\begin{pmatrix}\Gamma(s) & 0\\ 0 & \Gamma^{-1}(s)\end{pmatrix},$$

where $\Gamma(s)$ is an arbitrary analytic function regular at infinity and real-valued on the real axis.

To derive the integral equation containing the one and only sought function, we use the result obtained in Sec. 7.2, according to which there is an analytic continuation $\chi(t)$ of the matrix function $F(s)(1+\tau(s))\overset{\circ}{F}{}^{-1}(s)$, the latter being given on L, into domain D_- achieved by the Cauchy integral.

From the fact that

$$\chi(t)=F(s)(1+\tau(s))\overset{\circ}{F}{}^{-1}(s)$$

on L we find that

$$\Gamma^{-1}\chi_1{}^1 = (F_1{}^1\mathring{F}_2{}^2 - F_1{}^2\mathring{F}_2{}^1\Gamma^{-2})\,\tilde{\lambda},\ \tilde{\lambda} \equiv \lambda \exp(i\sigma), \tag{7.3.17}$$

$$\Gamma^{-1}\chi_1{}^2 = (-F_1{}^1\mathring{F}_1{}^2 + F_1{}^2\mathring{F}_1{}^1\Gamma^{-2})\,\tilde{\lambda}, \tag{7.3.18}$$

$$\Gamma\chi_2{}^1 = (\Gamma^2F_2{}^1\mathring{F}_2{}^2 - F_2{}^2\mathring{F}_2{}^1)\,\tilde{\lambda}, \tag{7.3.19}$$

$$\Gamma\chi_2{}^2 = (-\Gamma^2F_2{}^1\mathring{F}_1{}^2 + F_2{}^2\mathring{F}_1{}^1)\,\tilde{\lambda}. \tag{7.3.20}$$

We note that in D_- the functions $\Gamma^{-1}\chi_1{}^A$ and $\Gamma\chi_2{}^A$ $(A = 1, 2)$ are analytic, too.

Let us use the known Privalov solution (e.g. see [16]) for the inhomogeneous Riemann problem of finding functions $X_\pm$ analytic in D_+ and D_-, respectively, and such that

$$X_- = aX_+ + b$$

on L. If the index of $a(s)$ is zero, the Privalov solution has the form

$$X(t) = \exp(-F(t))\left(A - \frac{1}{2\pi i}\int b(s)\,[\exp F(s)]\frac{ds}{(s-t)}\right), \tag{7.3.21}$$

$X_- = X(t)$ at $t \in D_-$, $X_+ = X(t)$ at $t \in D_+$,

with $F(t) = (1/2\pi i)\int\{[\ln a(s)]/(s-t)\}\,ds$. Combining (7.3.21), (7.3.17), (7.3.18), and condition $F_A{}^B(0) = \delta_A^B$, we find that

$$F_1{}^1(t)\,\mathring{F}_2{}^2(t)\,\tilde{\lambda}(t) = 1 + \frac{1}{2\pi i}\int_L \tilde{\lambda}(s)\,F_1{}^2(s)\,\mathring{F}_2{}^1(s)\,\Gamma^{-2}(s)\,\frac{t\,ds}{s(s-t)}, \tag{7.3.17$'$}$$

$$F_1{}^1(t)\,\mathring{F}_1{}^2(t)\,\tilde{\lambda}(t) = \frac{1}{2\pi i}\int_L \tilde{\lambda}(s)\,F_1{}^2(s)\,\mathring{F}_1{}^1(s)\,\Gamma^{-2}(s)\,\frac{t\,ds}{s(s-t)}. \tag{7.3.18$'$}$$

Solving (7.3.17′) and (7.3.18′) for $F_1{}^1(t)$, we obtain a singular integral equation:

$$\mathring{Y}(t) = (1/2\pi i)\int_L Y(s)\,\Gamma^{-2}(s)\,W(s,t)\,ds, \tag{7.3.22}$$

where

$$\begin{aligned} sY(s) &= -F_1{}^2(s),\ s\mathring{Y}(s) = -\mathring{F}_1{}^2(s),\\ W(s,t) &= [\tilde{\lambda}(s)/(s-t)]\,[\mathring{F}_1{}^1(s)\,\mathring{F}_2{}^2(t) - \mathring{F}_2{}^1(s)\,\mathring{F}_1{}^2(t)]. \end{aligned} \tag{7.3.22$'$}$$

We can obtain a similar singular integral equation by solving both (7.3.19) and (7.3.20) for $F_2{}^2(t)$:

$$\overset{\circ}{Z}(t)=(1/2\pi i)\int Z(s)\,\Gamma^2(s)\,W(s,\ t)\,ds, \qquad (7.3.19')$$
$$Z(s)=F_2{}^1(s),\quad \overset{\circ}{Z}(s)=\overset{\circ}{F}{}_2{}^1(s).$$

Here $W(s,\ t)$ is again given by (7.3.22′).

The integral equations (7.3.22) and (7.3.19′) differ only by the power to which $\Gamma(s)$ is raised in the kernel.

If $w=+r$ (or $w=-r$), it is easy to find solutions to Eqs. (7.3.10) and (7.3.22) for arbitrary analytic functions $T(s)$ and $\Gamma(s)$. Indeed, in this case we have for the components $F_A{}^B(t)$ the following expression:

$$F_1{}^1=(1+2izt)^{-1},\ F_1{}^2=it\,(1+2izt)^{-1},\ F_2{}^1=0,\ F_2{}^2=1.$$

Whence $W(s,\ t)=(s-t)^{-1}$, and from (7.3.19′) and (7.3.22) it follows that

$$F_1{}^2(t)=it\,(1+2izt)^{-1}\Gamma^2(i/2z),$$
$$F_1{}^1(t)=(1+2izt)^{-1},\ F_2{}^1(t)=0,\ F_2{}^2(t)=1.$$

Hence, when $w=r$, the function $\mathscr{E}(r,\ q)=\Gamma^2(i/2r)$ is the sought solution. In view of the fact that $\Gamma(s)$ is chosen arbitrarily, $F_1{}^2(t)$ is an arbitrary analytic function regular at the origin of coordinates.

Let us assume that $\Gamma(s)$ is an arbitrary rational function that is bounded but nonzero at the point at infinity. We also assume that we can represent $\Gamma^{-2}(s)$ in the following form:

$$\Gamma^{-2}(s)=\prod_{k=1}^{N_1}(s-u_k)^{\alpha_k+1}/\alpha\prod_{n=1}^{N_2}(s-v_n)^{\beta_n+1}, \qquad (7.3.23)$$

with

$$\sum_{k=1}^{N_1}(\alpha_k+1)=\sum_{n=1}^{N_2}(\beta_n+1)=N.$$

When the denominator in (7.3.23) has complex conjugate roots, in the corresponding gravitational fields there appear ring singularities. If we integrate the right-hand side of (7.3.22) by parts and calculate the residues at points $s=v_n$, $n=1,\ 2,\ \ldots,\ N_2$, we obtain

$$\overset{\circ}{Y}(t)=Y(t)\,\Gamma^{-2}(t)+\sum_{n=1}^{N_2}\frac{1}{\beta_n!}\frac{\partial^{\beta_n}}{\partial s^{\beta_n}}$$
$$\times\Big[Y(s)\,W(s,\ t)\prod_{k=1}^{N_1}(s-u_k)^{\alpha_k+1}\Big/\prod_{k\neq n}^{N_2}(s-v_k)^{\beta_k+1}\Big]_{s=v_n}.$$

We simplify this expression by introducing the notations

$$\frac{\partial}{\partial s^m}\Big[Y(s)\prod_{k=1}^{N_1}(s-u_k)^{\alpha_k+1}\Big/\prod_{k\neq n}^{N_2}(s-v_k)^{\beta_k+1}\Big]_{s=v_n}$$

$$=Y_{n,m},\quad m=0,\ 1,\ \ldots,\ \beta_n;$$

$$\frac{1}{m!\,(\beta_n-m)!}\,\frac{\partial^{\beta_n-m}}{\partial s^{\beta_n-m}}W(s,\ t)\Big|_{s=v_n}=Q^{nm}(t),\quad n=1,\ 2,\ \ldots,\ N_2.$$

This yields

$$\overset{\circ}{Y}(t)=Y(t)\ \Gamma^{-2}(t)+\sum_{n=1}^{N_2}\sum_{m=0}^{\beta_n}Y_{nm}Q^{nm}(t). \qquad (7.3.24)$$

To find the sought solution for $\mathscr{E}$ we put $t=0$ in (7.3.23). This yields

$$\mathscr{E}=A\,[\overset{\circ}{\mathscr{E}}-i\sum_{n=1}^{N_2}\sum_{m=1}^{\beta_n}Y_{nm}Q^{nm}(0)],\ A=\alpha\prod_{n=1}^{N_2}v_n^{\beta_n+1}\Big/\prod_{k=1}^{N_1}u_k^{\alpha_k+1}. \qquad (7.3.25)$$

Hence, we need only to calculate the Y_{nm}. To this end we differentiate both sides of (7.3.24) l times with respect to t and then put $t=u_k$. This yields

$$\partial^l\overset{\circ}{Y}(u_k)/\partial t^l=\sum_{n=1}^{N_2}\sum_{m=0}^{\beta_n}Y_{nm}Q^{nm}_{kl},\quad Q^{nm}_{kl}=\partial^lQ^{nm}/\partial t^l|_{t=u_k}. \qquad (7.3.26)$$

Here $l=0,\ 1,\ \ldots,\ \alpha_k$, with k varying from 1 to N_1. Hence, Eqs. (7.3.26) comprise a set of N linear algebraic equations in N unknown Y_{nm} ($m=0,\ 1,\ \ldots,\ \beta_n$; $n=1,\ 2,\ \ldots,\ N_2$).

We now turn to the particular case of (7.3.23) where $\alpha_k=\beta_n=0$, i.e.

$$\Gamma^{-2}(s)=\prod_{k=1}^{N}(s-u_k)\Big/\prod_{k=1}^{N}(s-v_k), \qquad (7.3.27)$$

with u_k and v_k parameters that are never equal.

For the initial solution we take that for $\overset{\circ}{F}_A{}^B(t)$ corresponding to the metric $\overset{\circ}{g}_{11}=1$, $\overset{\circ}{g}_{12}=w$, $\overset{\circ}{g}_{22}=w^2-\rho^2$.

Suppose that

$$X_l=[iF_1{}^2(v_l)/v_l^2]\prod_{k=1}^{N}(v_l-u_k)\Big/\prod_{k\neq l}^{N}(v_l-v_k);$$

then, according to (7.3.26), the X_l satisfy a linear system:

$$1+\sum_{l=1}^{N}X_l/2+\sum_{l=1}^{N}\frac{\lambda_m^{+}+\lambda_l^{-}}{1/u_m-1/v_l}X_l=0, \qquad (7.3.28)$$

where

$$2\lambda_l^- \equiv \tilde{\lambda}(v_l)/v_l, \quad 2\lambda_m^+ = \tilde{\lambda}(u_m)/u_m,$$

$$\tilde{\lambda}(t) = \sqrt{(1-2tq)^2 + 4r^2t^2}\exp(i\sigma(t)),$$

$$\sigma(t) \equiv 2t\left[\int \frac{\partial w}{\partial \zeta}\frac{d\zeta}{(1+2i\zeta t)} + \int \frac{\partial w}{\partial \zeta^*}\frac{\partial \zeta^*}{(1-2i\zeta^* t)}\right].$$

If the constants u_k and v_l (k, $l = 1, \ldots, N$) are real-valued, it is expedient to introduce new constants according to the formulas

$$2u_k = 1/(q_k + m_k), \quad 2v_l = 1/(q_l - m_l).$$

This enables us to transform (7.3.27) into a linear set:

$$2 + \sum_{l=1}^{N} X_l[1 + (\lambda_k^+ + \lambda_l^-)/(q_k - q_l + m_k + m_l)] = 0, \tag{7.3.29}$$

where

$$\lambda_k^+ = \sqrt{(q_k + m_k - q)^2 + r^2}\exp(i\sigma_k^+),$$

$$\sigma_k^+ = \int \frac{\partial w}{\partial z}\frac{dz}{(q_k + m_k + iz)} + \int \frac{\partial w}{\partial z^*}\frac{dz^*}{(q_k + m_k - iz)},$$

$$\lambda_k^- = \sqrt{(q_k - m_k - q)^2 + r^2}\exp(i\sigma_k^-),$$

$$\sigma_k^- = \int \frac{\partial w}{\partial z}\frac{dz}{(q_k - m_k + iz)} + \int \frac{\partial w}{\partial z^*}\frac{dz^*}{(q_k - m_k - iz)}.$$

For the components of the generating matrix function $F(t)$ we have

$$\begin{aligned} F_1{}^1(t) &= S(t)/\tilde{\lambda}(t) - \sum_{l=1}^{N} X_l v_l/(v_l - t), \\ F_1{}^2(t) &= itS(t)\prod_{k=1}^{N}[(t - v_k)/(t - u_k)]/\tilde{\lambda}(t), \\ S(t) &\equiv 1 + \sum_{l=1}^{N} X_l[v_l(\tilde{\lambda}(t) + 2t\lambda_l^-) + v_l - t]/[2(v_l - t)], \end{aligned} \tag{7.3.30}$$

and

$$\begin{aligned} F_2{}^1(t) &= \prod_{k=1}^{N}[(t - u_k)/(t - v_k)]\; Q(t)/\tilde{\lambda}(t), \\ F_2{}^2(t) &= 1 + itQ(t) + it\sum_{l=1}^{N} Y_l u_l/(u_l - t), \\ Q(t) &\equiv w + iq + (1 - \tilde{\lambda}(t))/2it + \sum_{l=1}^{N} Y_l[u_l(2t\lambda_l^+ \\ &\quad + \tilde{\lambda}(t)) + t - u_l]/(t - u_l), \end{aligned} \tag{7.3.31}$$

where the functions Y_l can be found from the linear algebraic set

$$2\left(w+i\left(q-q_k\right)+i\lambda_k^- + im_k\right) + \sum_{l=1}^{N}\left[\frac{\lambda_k^- + \lambda_l^+}{q_l-q_k+m_k+m_l}+1\right]Y_l=0. \tag{7.3.32}$$

The solutions (7.3.30) and (7.3.31) correspond to the solution for N black holes lying on a symmetry axis in an arbitrary neutrino field.[7]

To find the sought components $H_A{}^B$, we can use the formula $\partial F_A{}^B/\partial t|_{t=0} = -iH_A{}^B$.

We then have

$$\begin{aligned}
&H_{11}=A\Big(1+\sum_{l=1}^{N}X_l\Big), \qquad A=\prod_{n=1}^{N}\left(v_n/u_n\right),\\
&H_{12}=2\left(w+iz\right)+\sum_{l=1}^{N}X_l\left[w+i\left(q-q_l\right)+i\lambda_l^- + im_l\right];\\
&H_{21}=\sum_{l=1}^{N}Y_l,\\
&H_{22}=A^{-1}\Big\{w^2-r^2+2iqw+2\int z^*\frac{\partial w}{\partial z^*}\,dz^*-2\int z\frac{\partial w}{\partial z}\,dz\\
&\qquad+2\sum_{l=1}^{N}Y_l\left[w+i\left(q-q_l\right)-\lambda_l^+ - im_l\right]\Big\}.
\end{aligned} \tag{7.3.33}$$

Note that when there is only one black hole, the explicit form of the solution is

$$\begin{aligned}
&H_{11}=\frac{\lambda^+ + \lambda^- - 2m}{\lambda^+ + \lambda^- + 2m},\\
&H_{12}=2\left(w+iq\right)-\frac{4m\left[w+i\left(q-q_1\right)+i\lambda^- + im\right]}{\lambda^+ + \lambda^- + 2m},\\
&H_{21}=-\frac{4m\left[w+i\left(q-q_1\right)+i\lambda^- + im\right]}{\lambda^+ + \lambda^- + 2m},\\
&H_{22}=\overset{\circ}{H}_{22}-\frac{8m\left[w+i\left(q-q_1\right)+i\lambda^- + im\right]\left[w+i\left(q-q_1\right)-i\lambda^+ - im\right]}{\lambda^+ + \lambda^- + 2m}.
\end{aligned} \tag{7.3.34}$$

In the absence of neutrino fields the solution given by (7.3.34) coincides with the Schwarzschild solution. Indeed, if we go over to the variables

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[7] In the absence of neutrino fields our solution becomes the N soliton solution of Belinskii and Zakharov [17].

$$2\,(R - m) = \lambda^+ + \lambda^-, \quad 2m\cos\theta = \lambda^+ - \lambda^-$$

and isolate the real parts in (7.3.34), we obtain

$$g_{11} = (1 - 2m/R), \quad g_{22} = -R^2\sin\theta, \quad g_{12} = 0.$$

7.4 ROTATION OF THE POLARIZATION VECTOR OF GRAVITATIONAL WAVES IN A BURST OF NEUTRINO RADIATION

Let us consider propagation of discontinuities of the first order of gravitational and electromagnetic waves in a neutrino field.

By the definition of a Hadamard discontinuity of the first order, the metric on the discontinuity surface remains continuous while the discontinuities in the first derivatives of the metric coefficients cannot be eliminated by an appropriate choice of the discontinuities in the second derivatives of the coordinate transitions. The coordinate transformations $x^{i'} = f^i\,(x^j)$ admit of discontinuities in the second derivatives:

$$[\partial^2 f^m/\partial x^i \partial x^k] = \gamma^{m'} l_i l_k, \tag{7.4.1}$$

where $l_i = \partial u/\partial x^i|_{u=0}$ and $u\,(x^k) = 0$ is the equation for the discontinuity surface.

The Hadamard conditions for the discontinuities in the first derivatives of the metric coefficients are

$$\partial g_{ij}/\partial x^k = l_k \zeta_{ij}. \tag{7.4.2}$$

These imply that when we transform to the reference frame $x^{i'}$, the jumps ζ_{ij} change in the following manner:

$$(\partial f^k/\partial x^j)\,(\partial f^l/\partial x^i)\,\zeta'_{kl} = \zeta_{ij} + l_i\gamma_j + l_j\gamma_i.$$

We can obtain algebraic conditions on ζ_{ij} either by integrating both sides of the Einstein equations over a layer of small thickness ε surrounding the surface $u = 0$ for continuous solutions that rapidly change across $u = 0$ and then going to the limit with $\varepsilon \to 0$, or by using a variational condition concerning the continuity of the generalized momentum on the discontinuity surface (see Sedov [18]):

$$[(\partial R l_k)/\partial\,(\partial g_{ij}/\partial x^k)] = 0,$$

whence we obtain

$$l^k l_k \zeta_{ij} - l_i l^k(\zeta_{kj} - g_{kj}\zeta^m_m/2) - l_j l^k\,(l_{ki} - g_{ki}\zeta^m_m/2) = 0.$$

These conditions imply that the discontinuities in the derivatives of the metric coefficients cannot be removed only on isotropic surfaces, where $l^k l_k = g^{ij}u_{,i}u_{,j} = 0$, and that

$$l^k(\zeta_{kj} - \zeta^m_m g_{kj}/2) = 0. \qquad (7.4.3)$$

Let us have corresponding to the discontinuity surface $u = 0$ a tetrad field l_i, m_i, m^*_i, n_i, of which the first three touch the surface $u = 0$ and are translated in parallel along the isotropic geodesics l_i. In the vicinity of surface $u = 0$ we build the tetrad field by parallel translation along vector n_i, which is assumed to be tangent to the family of isotropic geodesics that go transversally through the surface $u = 0$ touching the field of vectors n_i on the surface $u = 0$ defined above. We note that the integral curves of vector l_i are geodesics only on the discontinuity surface.

Then, decomposing ζ_{ij} along the tetrad vectors of (7.4.3), we obtain

$$\zeta_{ij} = Pm_im_j + P^*m^*_im^*_j + l_i\tilde{\gamma}_j + l_j\tilde{\gamma}_i,$$

where the $\tilde{\gamma}_i$ are the components of an arbitrary vector on the surface $u = 0$.

The Ricci rotation coefficients of an isotropic tetrad, except λ, are in this case continuous on the discontinuity surface and

$$[\lambda] = -P/2, \quad \lambda \equiv -m^{*i}m^{*j}\nabla_i n_j.$$

According to the definition of the Ricci tensor component Φ_{20}, we have

$$[\Phi_{20}] = -(1/2)(DP + \nabla_i l^i P/2). \qquad (7.4.4)$$

The Maxwell equations provide the algebraic conditions on the discontinuities of the tetrad components of electromagnetic field bivector: $[F_{ij}]\, l^j = 0$. Whence

$$[F_{ij} + (i/2)\,\varepsilon_{klij}F^{kl}] = 2fl_{[i}m_{j]},$$

where ε_{klij} is the totally antisymmetric symbol in four dimensions. We can easily calculate the jump in the Φ_{20} component of the energy-momentum tensor of the electromagnetic field:

$$[\Phi_{20}] = \frac{\varkappa}{4\pi}\Phi^*_0 f \qquad (7.4.5)$$

(here $\varkappa$ is the Einstein gravitational constant). Subtracting from each other the Maxwell equations written for both sides of the discontinuity surface, we obtain

$$Df + \nabla_i l^i f/2 - \Phi_0 P/2 = 0, \qquad (7.4.6)$$

where Φ_0 is the tetrad component of the electromagnetic field component continuous on the discontinuity.

Let us now turn to the Weyl equations. They imply that the only component of the neutrino field that experiences a discontinuity is

Φ, while the other component Ψ must remain continuous. But the discontinuity in Φ is not connected with the discontinuity in the gravitational field, and therefore from now on we will assume that the discontinuity is zero thanks to the initial conditions.

The discontinuity in the component Φ_{20} of the energy-momentum tensor of the neutrino field can easily be calculated:

$$[\Phi_{20}^{(\nu)}] = -i\varkappa P\Psi\Psi^*/2. \tag{7.4.7}$$

Whence, from (7.4.4), (7.4.5), and (7.4.7) we finally obtain

$$0 = DP + \nabla_i l^i P/2 + (4G/c^4)\, f\Phi_0^* + (8\pi G/c^4)\, iP\Psi\Psi^*, \tag{7.4.8}$$

where G is the gravitational constant. The set of equations (7.4.6) and (7.4.8) is a closed set of ordinary equations describing the time behavior of the discontinuity in the gravitational and electromagnetic fields. It is interesting that the presence of a neutrino field has no effect on the total size of the discontinuities in the gravitational and electromagnetic fields. Indeed, from (7.4.6) and (7.4.8) it follows that

$$\nabla_i\, [l^i\, ((8G/c^4)\, |f^2| + |P^2|)] = 0.$$

Suppose there is no external electromagnetic field. Then Eq. (7.4.8) becomes closed and it yields

$$P\sqrt[4]{-g} = \text{const}\cdot\exp\Big[-(i8\pi G/c^4)\int_{\alpha_0}^{\alpha}\Psi\Psi^*\, d\alpha\Big)\Big]. \tag{7.4.9}$$

Here g is the determinant of the metric tensor in an isotropic-geodesic coordinate system.

The last equation implies that the polarization vector of the gravitational waves rotates in the neutrino field. Indeed, the argument of the complex function P characterizes the angle between the polarization vector of the gravitational wave and the initial polarization vector translated parallelly along the ray. The greater the intensity of the neutrino field, the greater the angle through which the gravitational wave polarization vector is rotated (see formula (7.4.9)).

In the presence of external electromagnetic fields, this effect is imposed on the effect discovered by the author [13] of successive transmutation of the discontinuities in the gravitational and electromagnetic fields.

Let us put $\Phi_0 = a\exp(ib)$, where a and b are real-valued functions. To determine the changes along rays, let us go over to an isotropic-geodesic system of coordinates and new variables

$$\mathcal{P} = P\sqrt[4]{-g}\exp(ib), \quad \mathcal{F} = 2\sqrt{G}\, f\sqrt[4]{-g}/c^2, \quad 2\sqrt{G}\, a/c^2 \equiv \Phi.$$

Then the set of equations (7.4.6) and (7.4.8) can be written thus:

$$d\mathcal{P}/d\alpha + \Phi\mathcal{F} + i\,[(8\pi G/c^4)\,\Psi\Psi^* - \partial b/\partial\alpha]\,\mathcal{P} = 0,$$

$$d\mathcal{F}/d\alpha - \mathcal{P}/2 = 0.$$

This set implies that the effect of the neutrino field on the time behavior of the discontinuities is equivalent to the action of an external electromagnetic field Φ_0^* with the same intensity a but with a different law of variation of the argument along the ray: $\Phi_0^* = a \exp(ib^*)$, with

$$b^* = b - (8\pi G/c^4)\int_{\alpha_0}^{\alpha} \Psi\Psi^*\,d\alpha.$$

Therefore, the effect of successive transmutation of the gravitational and electromagnetic discontinuities in a neutrino electrovacuum is accompanied by a consistent rotation of the polarization vector of these waves, i.e. the situation is the same as in an arbitrary electrovacuum case [13].

The above results also hold for the case of short waves propagating in neutrino-electromagnetic fields.

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8

Tensor Representation of Spinor Fields

V. A. Zhelnorovich, *Cand. Sc. (Physics and Mathematics)*
Institute of Mechanics, M. V. Lomonosov Moscow State University

INTRODUCTION

As we know, finite dimensional linear representations of orthogonal groups are exhausted by tensor and spinor representations. Hence, the geometric objects associated with orthogonal group representations are also exhausted by tensors and spinors of various ranks. The theory of tensors and tensor analysis are widely used in modern physics: both are tools of modern physical theories. The theory of spinors is now used largely in field theory and quantum mechanics (to mention the physical applications). Lately the notion of a spinor and spinor analysis are also being used (e.g. see [1-3]) in the mechanics of continuous media, for one, to build models of magnetizable and polarizable media with a microstructure.

The present article presents the methods of tensor representation of spinors in three- and four-dimensional spaces. As an example we consider the tensor form of some spinor equations, including important equations of field theory. The developed approach is also used in the relativistic hydrodynamics of magnetizable fluids with intrinsic angular momentum to build solutions of the differential equations.

The main results connected with the tensor representation of spinors consist in establishing a one-to-one invariant algebraic and geometric interrelationship (see [1, 2, 4]) between spinors and some simple tensors whose components are connected by bilinear algebraic equations.[1] This interrelationship between tensors and spinors enables writing spinor equations in the form of equivalent tensor equations, which is important and useful in physical applications.

8.1 DIRAC MATRICES

The spinor representation of pseudo-orthogonal groups and spinors in a four-dimensional pseudo-Euclidean space can be defined by means of an algebra generated by matrices γ_i defined below. Let the

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[1] The problem of representing spinors as objects of a mechanical or geometric nature was posed by L. I. Sedov in 1964 at the All-Union Conference on Mechanics Workshop "Interesting problems and methods of fluid mechanics" [5]. It was completely solved in [1, 2, 4], where the author specified the tensors that fully define a spinor as an invariant geometric object in any, in general curvilinear, system of coordinates.

γ_i $(i = 1, 2, 3, 4)$ be 4×4, in general complex, matrices that satisfy the equation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij}I, \tag{8.1.1}$$

where I is the unit 4×4 matrix; the components g_{ij} form a diagonal matrix:

$$g_{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{8.1.2}$$

Matrices that satisfy (8.1.1) are called Dirac matrices. With the help of the four γ_i we introduce four matrices $\overset{*}{\gamma}{}^i$, six matrices $\gamma_{ij} = -\gamma_{ji}$, and the matrix γ^5:

$$\gamma_{ij} = (1/2)\,(\gamma_i\gamma_j - \gamma_j\gamma_i), \quad \overset{*}{\gamma}{}^i = -\,(1/6)\,\varepsilon^{ijks}\gamma_j\gamma_k\gamma_s,$$
$$\gamma^5 = -(1/24)\,\varepsilon^{ijks}\gamma_i\gamma_j\gamma_k\gamma_s, \tag{8.1.3}$$

where ε^{ijks} is the totally antisymmetric symbol in four dimensions (the Levi-Civita symbol; $\varepsilon^{1234} = -1$). In Eqs. (8.1.3) and in the material that follows (except if stated otherwise) we assume Einstein's summation convention to be valid, i.e. any term in which an index appears twice will stand for the sum of all such terms as the index assumes all of a preassigned range of values.

The main property of a Dirac matrix is formulated by Pauli's theorem: any two systems of matrices γ_i and $\tilde{\gamma}_i$ satisfying Eq. (8.1.1) are always related thus:

$$\tilde{\gamma}_i = T^{-1}\gamma_i T, \tag{8.1.4}$$

where T is a nonsingular 4×4 matrix determined to within a factor that is an arbitrary nonzero complex number.

If the γ_i are the solution of Eqs. (8.1.1), then, obviously, the transpose γ_i^{T} and the Hermitian conjugate $\dot{\gamma}_i^{\mathrm{T}}$ (T is the transposition symbol, and the dot over a symbol stands for complex conjugation) are also solutions of Eqs. (8.1.1). Pauli's theorem then implies that there exist, to within factors that are nonzero complex numbers, nonsingular matrices E and β such that[2]

$$\gamma_i^{\mathrm{T}} = -E\gamma_i E^{-1}, \quad \dot{\gamma}_i^{\mathrm{T}} = -\beta\gamma_i\beta^{-1}. \tag{8.1.5}$$

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[2] In the physical literature (e.g. see [6]) the matrix E^{-1} is usually denoted by C (the operator of charge conjugation in Dirac's theory).

We can easily show that E is antisymmetric, while β can be taken Hermitian:

$$E^{\mathrm{T}} = -E, \quad \overset{\bullet}{\beta}{}^{\mathrm{T}} = \beta. \tag{8.1.6}$$

In what follows we will use the matrices γ_i in the form

$$\gamma_1 = \begin{Vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Vmatrix}, \quad \gamma_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix},$$

$$\gamma_3 = \begin{Vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{Vmatrix}, \quad \gamma_4 = \begin{Vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{Vmatrix}. \tag{8.1.7}$$

Then

$$E = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \quad \beta = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, \quad \gamma^5 = \begin{Vmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{Vmatrix}. \tag{8.1.8}$$

8.2 THE SPINOR REPRESENTATION OF THE LORENTZ GROUP

Let us take the four-dimensional real pseudo-Euclidean vector space E_4 of signature $(+, +, +, -)$ with an orthonormal basis Э_i, $i = 1, 2, 3, 4$. The scalar products of the basis vectors, $g_{ij} = (\text{Э}_i, \text{Э}_j)$, in E_4 are determined by matrix (9.1.2). Consider the full homogeneous Lorentz group of linear transformations of basis vectors in E_4:

$$\text{Э}'_i = l^j{}_i \text{Э}_j. \tag{8.2.1}$$

The transformation coefficients $l^j{}_i$ of (8.2.1) satisfy the equation

$$l^j{}_i l^m{}_n g_{jm} = g_{in}, \tag{8.2.2}$$

which may serve as a definition of a full homogeneous Lorentz transformation.

The full homogeneous Lorentz group can be divided into four connected components: proper Lorentz transformations and improper Lorentz transformations of the 1st, 2nd, and 3rd kinds; all are defined by the signs of $l^4{}_4$ and of the determinant $\Delta = \det \| l^j{}_i \|$ according to the table:

transformations	proper	1st kind	2nd kind	3rd kind
Δ	+	−	−	+
$l^4{}_4$	+	+	−	−

Improper Lorentz transformations of the 1st, 2nd, and 3rd kinds can be represented in the form of products of the proper Lorentz transformation and one of the reflection transformations P, T, or J given by the matrices

$$P = \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad T = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix},$$

$$J = \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

(i) *Spinor representation of the proper Lorentz group.* Let us consider the Dirac matrices γ_i, which satisfy Eq. (8.1.1). The orthogonality condition (8.2.2) implies that the matrices $\gamma'_i = l^j{}_i\gamma_j$ also satisfy Eq. (8.1.1). From Pauli's theorem it then follows that there exists a 4 × 4 matrix S (defined to within a factor that is an arbitrary nonzero complex number) such that

$$l^j{}_i\gamma_j = S^{-1}\gamma_i S. \tag{8.2.3}$$

Transposing Eq. (8.2.3) and using (8.1.5) for γ_i^{T}, we obtain

$$l^j{}_i\gamma_j = E^{-1}S^{\mathrm{T}}E\gamma_i\,(E^{-1}S^{\mathrm{T}}E)^{-1}.$$

Comparing this equation with Eq. (8.2.3), we find

$$\gamma_i SE^{-1}S^{\mathrm{T}}E = SE^{-1}S^{\mathrm{T}}E\gamma_i.$$

Hence, the matrix $SE^{-1}S^{\mathrm{T}}E$ commutes with all the γ_i and, by the well-known Shur theorem, is a multiple of the unit matrix: $SE^{-1}S^{\mathrm{T}}E = \lambda I$, where λ is a nonzero complex number. We can also write this equation as $S^{\mathrm{T}}ES = \lambda E$. Since S is defined in Eq. (8.2.3) to within an arbitrary nonzero complex factor, by normalizing S

by the condition

$$S^{\mathrm{T}}ES = E \tag{8.2.4}$$

we find that Eq. (8.2.3) combined with (8.2.4) relates each proper Lorentz transformation L to two matrices, S and $-S$.

We think of S and $-S$ as identical transformations and each pair of transformations S and $-S$ we consider as one element $\pm S$. The product is defined on the set of all pairs $\pm S$ as

$$(\pm S_1)(\pm S_2) \overset{\mathrm{def}}{=} \pm (S_1 S_2), \tag{8.2.5}$$

where $S_1 S_2$ is the usual product of matrices S_1 and S_2. We can easily see that the set $\{\pm S\}$, corresponding to the proper Lorentz group, is a group with respect to multiplication (8.2.5), which realizes the representation of the proper Lorentz group known as the spinor representation[3]. The group $\{\pm S\}$ will be called the group of spinor transformations.

(ii) *Spinor representations of the full Lorentz group.* It is expedient to break up the spinor representations of the homogeneous Lorentz group into several classes [1, 2]. The first class of spinor representations is defined by the equations

$$l^j{}_i \gamma_j = S^{-1} \gamma_i S \tag{8.2.6}$$

and one of the following normalization conditions:

$$\begin{aligned} &\text{(a)}\ S^{\mathrm{T}}E\,S = E,\\ &\text{(b)}\ S^{\mathrm{T}}E\,S = E \operatorname{sgn} \Delta,\\ &\text{(c)}\ S^{\mathrm{T}}E\,S = E \operatorname{sgn} (\Delta l^4{}_4),\\ &\text{(d)}\ S^{\mathrm{T}}E\,S = E \operatorname{sgn} l^4{}_4. \end{aligned} \tag{8.2.7}$$

For all normalization conditions (8.2.7) and with Eqs. (8.2.6) valid we have

$$\dot{S}^{\mathrm{T}}\beta S = \beta \operatorname{sgn} l^4{}_4, \tag{8.2.8}$$

which connects the spinor transformations S and the complex conjugate transformations $\dot{S}$.

We write out the spinor transformations S_P, S_T, and S_J corresponding to the reflection transformations P, T, and J for various normalization conditions (8.2.7), assuming that the matrices E and

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[3] Obviously, if the set of some matrices $\{S\}$ forms a group, the corresponding set $\{\pm S\}$ also forms a group with respect to multiplication (8.2.5) (which is the factor group $S/\pm I$).

γ_i are defined by (8.1.7) and (8.1.8):

	a	b	c	d
S_P	γ_4	$i\gamma_4$	$i\gamma_4$	γ_4
S_T	$\overset{*}{i\gamma_4}$	$\overset{*}{\gamma_4}$	$\overset{*}{i\gamma_4}$	$\overset{*}{\gamma_4}$
S_J	$i\gamma^5$	$i\gamma^5$	γ^5	γ^5

For all normalization conditions (8.2.7), the spinor transformations S_P, S_T, and S_J anticommute:

$$S_P S_T = -S_T S_P,\quad S_P S_J = -S_J S_P,\quad S_T S_J = -S_J S_T.$$

The second class of spinor representations is defined by the equations

$$l^j{}_i \gamma_j = S^{-1}\gamma_i S \operatorname{sgn} \Delta \tag{8.2.9}$$

and one of the normalization conditions in (8.2.7).

For all normalization conditions (8.2.7) and with Eqs. (8.2.9) valid we have

$$S^{\mathrm{T}}\beta S = \beta \operatorname{sgn}(\Delta l^4{}_4).$$

We write out the spinor transformations S_P, S_T, and S_J of this class corresponding to the reflection transformations P, T, and J for various normalization conditions (8.2.7), assuming that the matrices E and γ_i are defined by (8.1.7) and (8.1.8):

	a	b	c	d
S_P	$\overset{*}{i\gamma_4}$	$\overset{*}{\gamma_4}$	$\overset{*}{\gamma_4}$	$\overset{*}{i\gamma_4}$
S_T	γ_4	$i\gamma_4$	γ_4	$i\gamma_4$
S_J	$i\gamma^5$	$i\gamma^5$	γ^5	γ^5

For all normalization conditions (8.2.7), the spinor transformations S_P, S_T ,and S_J of the second class anticommute, too.

The spinor representations $L \to \{\pm S_{\mathrm{II}}\}$ defined by Eqs. (8.2.9) and (8.2.7) are equivalent to the spinor representations $L \to \{\pm S_{\mathrm{I}}\}$

defined by Eqs. (8.2.6) and (8.2.7) and the two are related thus: $\pm S_{\mathrm{II}} = A\,(\pm S_{\mathrm{I}})\,A^{-1}$, with $A = I + \gamma^5$.

The third class of spinor representations of the full Lorentz group is defined by the equations

$$l^j{}_i\gamma_j = S^{-1}\gamma_i S\ \mathrm{sgn}\,(\Delta l^4{}_4) \tag{8.2.10}$$

and one of the normalization conditions in (8.2.7).

In view of Eqs. (8.2.10) we also have

$$\dot{S}^{\mathrm{T}}\beta S = \beta\ \mathrm{sgn}\,\Delta.$$

Assuming that the matrices γ_i and E are defined by (8.1.7) and (8.1.8), we write out the spinor representations S_P, S_T, and S_J of this class corresponding to the reflection transformations:

	a	b	c	d
S_P	$\overset{*}{i\gamma_4}$	$\overset{*}{\gamma_4}$	$\overset{*}{\gamma_4}$	$\overset{*}{i\gamma_4}$
S_T	$\overset{*}{i\gamma_4}$	$\overset{*}{\gamma_4}$	$\overset{*}{i\gamma_4}$	$\overset{*}{\gamma_4}$
S_J	I	I	iI	iI

For all normalization conditions (8.2.7), the spinor transformations S_P, S_T, and S_J of the third class commute:

$$S_PS_T = S_TS_P,\quad S_PS_J = S_JS_P,\quad S_TS_J = S_JS_T.$$

Finally, the fourth class of spinor representations is defined by the equations

$$l^j{}_i\gamma_j = S^{-1}\gamma_i S\ \mathrm{sgn}\ l^4{}_4 \tag{8.2.11}$$

and one of the normalization conditions in (8.2.7).

For all normalization conditions (8.2.7) and in view of Eqs. (8.2.11) the following is also true:

$$\dot{S}^{\mathrm{T}}\beta S = \beta.$$

If the matrices E and γ_i are defined by Eqs. (8.1.7) and (8.1.8), for the spinor representations S_P, S_T, and S_J of this class corresponding to the reflection transformations P, T, and J we have the following table:

	a	b	c	d
S_P	γ_4	$i\gamma_4$	$i\gamma_4$	γ_4
S_T	γ_4	$i\gamma_4$	γ_4	$i\gamma_4$
S_J	I	I	iI	iI

For all normalization conditions (8.2.7), the spinor transformations of the fourth class commute, too.

The spinor representations $L \rightarrow \{\pm S_{\mathrm{IV}}\}$ defined by Eqs. (8.2.11) and (8.2.7) are equivalent to the spinor representations $L \rightarrow \{\pm S_{\mathrm{III}}\}$ defined by Eqs. (8.2.10) and (8.2.7), and the two are related thus: $\pm S_{\mathrm{IV}} = A\,(\pm S_{\mathrm{III}})\,A^{-1}$, with $A = I + \gamma^5$.

8.3 SPINORS IN FOUR-DIMENSIONAL PSEUDO-EUCLIDEAN VECTOR SPACE

Let S_4 be a four-dimensional complex linear (vector) space and let ε_A be a basis in S_4. Take in S_4 an arbitrary vector $\psi^A\varepsilon_A$ and another vector $-\psi^A\varepsilon_A$ defined in the basis ε_A, respectively, by components ψ^A and $-\psi^A$. We think of $\psi^A\varepsilon_A$ and $-\psi^A\varepsilon_A$ as identical vectors and to each pair of vectors $\psi^A\varepsilon_A$ and $-\psi^A\varepsilon_A$ we take as one object $\psi = \pm\psi^A\varepsilon_A$ in S_4.[4]

We also think of the bases ε_A and $-\varepsilon_A$ of space S_4 as identical, and the same for the components ψ^A and $-\psi^A$. The pair of bases ε_A and $-\varepsilon_A$ we consider as one element $\pm\varepsilon_A$ and the pair ψ^A and $-\psi^A$ as one element $\pm\psi^A$. To each pair of bases $\pm\varepsilon_A$ and each pair of component systems $\pm\psi^A$ we assign in a one-to-one manner the pair of vectors $\pm\psi^A\varepsilon_A$ in space S_4.

Now let us take a group of linear transformations $S = \| S^B{}_A \|$ of bases ε_A in vector space S_4. Suppose $\{\pm S\}$ is the group whose elements are pairs of transformations S and $-S$. By means of the $\{\pm S\}$ group, on the set of pairs of bases $\pm\varepsilon_A$ in S_4 and on the set of pairs of components $\pm\psi^A$ we define the transformations $\pm\varepsilon_A \rightarrow$

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[4] Geometrically, such an object defines an unoriented segment in S_4.

$\pm\varepsilon'_A$ and $\pm\psi^A \to \pm\psi'^A$ by the following relationships:

$$\begin{aligned} &\pm\,\varepsilon'_A = (\pm\, Z^B{}_A)\,(\pm\,\varepsilon_B) \overset{\text{def}}{=\!=} \pm\, Z^B{}_A\varepsilon_B, \\ &\pm\,\psi'^A = (\pm\, S^A{}_B)\,(\pm\,\psi^B) \overset{\text{def}}{=\!=} \pm\, S^A{}_B\psi^B, \end{aligned} \tag{8.3.1}$$

with $\| Z^B{}_A \| = S^{-1}$.

Obviously, under such transformations the pair of vectors $\pm\psi^A\varepsilon_A$, corresponding to the pairs $\pm\psi^A$ and $\pm\varepsilon_A$, are invariant:

$$\psi = \pm\psi^A\varepsilon_A = \pm\psi'^A\,\varepsilon'_A. \tag{8.3.2}$$

Consider the case where $\{\pm S\}$ is the spinor group that effects the representation of the full Lorentz group of transformations of the bases of the four-dimensional pseudo-Euclidean space E_4. To an orthonormal basis $Э_i$ in E_4 we assign a pair of bases $\pm\varepsilon_A$ in S_4, while to each orthonormal basis $Э'_i$ in E_4 obtained from $Э_i$ via a Lorentz transformation L by (8.2.1) we assign a pair of bases $\pm\varepsilon'_A$ in S_4 obtained from $\pm\varepsilon_A$ via the transformation

$$\pm\varepsilon'_A = \pm Z^B{}_A\varepsilon_B, \tag{8.3.3}$$

with $\| Z^B{}_A \| = S^{-1}$.

According to the established correspondence between E_4 and S_4, under the Lorentz transformation (8.2.1) of the $Э_i$ bases in E_4, the pairs of bases $\pm\varepsilon_A$ in S_4 undergo the transformation S^{-1}. After we have established the correspondence between E_4 and S_4, we can consider the object $\psi = \pm\psi^A\varepsilon_A$ as an invariant object in the pseudo-Euclidean space E_4, and when the transformation $Э'_i = l^j{}_iЭ_j$ in E_4 is effected, the components $\pm\psi^A$ transform in the following manner:

$$\pm\psi'^A = \pm S^A{}_B\psi^B. \tag{8.3.4}$$

The invariant object $\psi = \pm\psi^A\varepsilon_A$, where the pairs of contravariant components $\pm\psi^A$ and the pairs of bases $\pm\varepsilon_A$ are taken respective to an orthonormal basis $Э_i$ of the pseudo-Euclidean space E_4 and transform according to (8.3.3) and (8.3.4) under the Lorentz transformation of the basis $Э_i$, is called a spinor of rank 1 in the pseudo-Euclidean space E_4. The bases $\pm\varepsilon_A$ are commonly known as spin-bases. According to the definition, the components ψ^A and $-\psi^A$ referred to one spin-basis $\pm\varepsilon_A$ define one and the same spinor ψ.

The covariant components ψ_B of a spinor are given by the relationship

$$\psi_B = e_{BA}\psi^A, \tag{8.3.5}$$

where e_{BA} are the components of the matrix E defined in (8.1.5). If we denote the components of the inverse E^{-1} by e^{BA}, then

$$\psi^A = e^{AB}\psi_B. \tag{8.3.6}$$

We will call objects whose components contain both spinor and tensor indices spin-tensors. Equations (8.2.6) imply that the Dirac matrices γ_i determine components (invariant under transformations of the basis $Э_i$) of a spin-tensor with one covariant tensor index and one covariant and one contravariant spinor indices. For the spinor representations defined by Eqs. (8.2.9)-(8.2.11), the γ_i are invariant under continuous transformations of the $Э_i$ basis. From (8.2.7) it follows that the components of $E = \| e_{BA} \|$ (with $E^{-1} = \| e^{BA} \|$) are covariant (contravariant) components of a second-rank spinor that are invariant at least under continuous Lorentz transformations of the $Э_i$ basis.

Raising and lowering the indices of a spinor of arbitrary rank is achieved by using the components of the invariant spinor E according to (8.3.5) and (8.3.6). Note that in both (8.3.5) and (8.3.6) contraction is done with respect to the second index in e_{BA} and e^{BA} (which is essential for spinors of even rank).

8.4 CONJUGATE SPINORS

Besides spinors that transform via matrices S, we can introduce geometric objects that transform via matrices S, which are complex conjugates of S. We will label the components of these new objects by a dot and call them dotted components. It is clear that the complex conjugate components $\dot{\psi}_B$ and $\dot{\psi}^A$ of a spinor transform like the components with dotted indices $\psi_{\dot{B}}$ and $\psi^{\dot{A}}$.

From Eqs. (8.2.8) (or similar equations for other classes of spinor representations) it follows that the matrix β determines a second-rank spinor invariant under continuous Lorentz transformations and has the following index structure: $\beta = \| \beta_{\dot{B}A} \|$.

A second-rank spinor determined by the covariant components

$$\psi^+_A = \beta_{\dot{B}A}\psi^{\dot{B}} \tag{8.4.1}$$

is said to be conjugate to a spinor with components ψ^B calculated in the same basis as the ψ^+_A. For the contravariant components ψ^{+B} of the conjugate spinor we have

$$\psi^{+B} = e^{BA}\psi^+_A = e^{BA}\beta_{\dot{C}A}\psi^{\dot{C}}. \tag{8.4.2}$$

It can be easily shown that in the four-dimensional space E_4, conjugation of the conjugate spinor $\boldsymbol{\psi}^+$ yields $\boldsymbol{\psi}$:

$$(\psi^{+A})^+ = \psi^A. \tag{8.4.3}$$

8.5 THE RELATION BETWEEN EVEN-RANK SPINORS AND TENSORS

As is known (see [7]), the 16 matrices I, γ_i, γ_{ij} $(i < j)$, $\overset{*}{\gamma}_i$, and γ^5 are linearly independent and form a basis in the space of 4×4 complex matrices. For this reason the components ψ^{BA} of any second-rank spinor can be written as

$$\psi^{BA} = (1/4)\left[-Fe^{BA} + F^i\gamma_i^{BA} + (1/2)\,F^{ij}\gamma_{ij}^{BA} + \overset{*}{F}{}^i\overset{*}{\gamma}{}_i^{BA} + \overset{*}{F}\gamma^{5BA}\right], \quad (8.5.1)$$

with

$$\begin{aligned} &F = e_{BA}\psi^{BA}, \quad F^i = -\gamma^i_{BA}\psi^{BA}, \quad F^{ij} = \gamma^{ij}_{BA}\psi^{BA}, \\ &\overset{*}{F}{}^i = \overset{*}{\gamma}{}^i_{BA}\psi^{BA}, \quad \overset{*}{F} = -\gamma^5_{BA}\psi^{BA}. \end{aligned} \quad (8.5.2)$$

The spinor indices on the components of spin-tensor γ in Eqs. (8.5.1) and (8.5.2) are lowered or raised by using the spinor E. For instance,

$$\gamma^i_{BA} = e_{BC}\gamma^{Ci}{}_{A}, \quad \gamma_i^{BA} = e^{AC}\gamma^B_{iC}.$$

Under continuous Lorentz transformations, the quantities F and $\overset{*}{F}$ in Eqs. (8.5.1) and (8.5.2) are invariants, while F^i, F^{ij}, and $\overset{*}{F}{}^i$ are transformed via a tensor law.

It is clear that components of any even-rank spinor can be expanded in each pair of indices with respect to the set of spin-tensors I, γ_i, γ_{ij}, $\overset{*}{\gamma}_i$, and γ^5. For this reason even-rank spinors in E_4 are equivalent to collections of tensors in E_4.

8.6 THE RELATION BETWEEN FIRST-RANK SPINORS AND SYSTEMS OF COMPLEX TENSORS

Let us suppose that $L \to S$ is a representation of the Lorentz group of transformations of the pseudo-Euclidean space E_4 acting in the space of variables $a_1, a_2, \ldots, a_N$. We say that a geometric object is given in E_4 if in each orthonormal basis $Э_i$ of E_4 there is a system of components $a_1, a_2, \ldots, a_N$ that transforms according to the representation $L \to S$ when we go over from the orthonormal basis to a new orthonormal basis $Э'_i$ in the same space E_4.

Definition. *Two geometric objects A_1 and A_2 in E_4 are said to be equivalent if in each orthonormal basis $Э_i$ of E_4 there is a one-to-one correspondence between the components of A_1 and A_2, i.e. $A_1 = f(A_2)$, that is invariant with respect to the choice of basis $Э_i$.*

It is obvious that the above definition of equivalent objects is reflexive, symmetric, and transitive, and hence possesses all the properties of an equivalence relation.

According to this definition, if we know the components of a geometric object in every basis of E_4, we can unambigously calculate the components of an equivalent object. Hence, we can either use the components of A_1 or those of an equivalent object A_2. However, the components of two equivalent objects, may, according to the above definition, transform by essentially different representations.

Consider an arbitrary $r \times r$ complex matrix $\| \psi^{BA} \|$. Obviously, if the elements ψ^{BA} of the matrix are products of complex numbers, i.e.

$$\psi^{BA} = \psi^B \psi^A, \tag{8.6.1}$$

we have the following equations:

$$\psi^{BA} = \psi^{AB}, \quad \psi^{AB}\psi^{CD} = \psi^{AC}\psi^{BD} = \psi^{AD}\psi^{BC}. \tag{8.6.2}$$

Conversely, if the elements of matrix $\| \psi^{BA} \|$ satisfy Eqs. (8.6.2), there exists a system of r complex-valued components ψ^A defined to within sign and such that $\psi^{BA} = \psi^B \psi^A$. Indeed, if all the elements of $\| \psi^{BA} \|$ are zeros, then we put $\psi^A = 0$. If at least one element is not zero, we put

$$\psi^A = \psi^{BA}\eta_B / (\pm \sqrt{\psi^{CD}\eta_C\eta_D}), \tag{8.6.3}$$

where η_C $(C = 1, 2, \ldots, r)$ are arbitrary and, in general, complex numbers satisfying the condition $\psi^{CD}\eta_C\eta_D \neq 0$. In (8.6.3) we sum over the indices B, C, and D from 1 to r. We can easily see that if the ψ^{BA} in (8.6.3) satisfy Eqs. (8.6.2), the ψ^A defined by (8.6.3) satisfy Eq. (8.6.1).

In view of Eqs. (8.6.2), the definition (8.6.3) for ψ^A does not depend on the choice of the numbers η_C. Indeed, if $\psi^{CD}\eta_C\eta_D \neq 0$ and $\psi^{CD}\eta^*_C\eta^*_D \neq 0$, employing Eqs. (8.6.2) we obtain

$$\begin{aligned}\frac{\psi^{BA}\eta_B}{\pm\sqrt{\psi^{CD}\eta_C\eta_D}} &= \frac{\psi^{MN}\eta^*_M\eta^*_N\psi^{BA}\eta_B}{\pm\sqrt{(\psi^{MN}\eta^*_M\eta^*_N)^2\,\psi^{CD}\eta_C\eta_D}} \\ &= \frac{\psi^{NA}\eta^*_N\psi^{BM}\eta_B\eta^*_M}{\pm\sqrt{\psi^{MN}\eta^*_M\eta^*_N\,(\psi^{CD}\eta_C\eta^*_D)^2}} = \frac{\psi^{NA}\eta^*_N}{\pm\sqrt{\psi^{MN}\eta^*_M\eta^*_N}}.\end{aligned} \tag{8.6.4}$$

If in (8.6.3) we define the η_C by the equalities $\eta_C = \delta_{BC}$, where B is a fixed number, and δ_{BC} is the Kronecker delta (i.e. all the η_C expect one is zero, and the one numbered B is unity), we can write (8.6.3) in a simpler form:

$$\psi^A = \psi^{BA} / (\pm \sqrt{\psi^{BB}}), \tag{8.6.5}$$

where we do not sum over B. Since definition (8.6.3) does not depend on the η_C, formulas (8.6.3) and (8.6.5) determine the same dependence of ψ^A on ψ^{BA}.

If the components ψ^{BA} in (9.6.3) are transformed according to the formula

$$\psi^{BA} \to (\psi^{BA})' = S^B{}_C S^A{}_M \psi^{CM},$$

where the $S^B\ \xi_C$, in general, form an arbitrary $r \times r$ complex (nonsingular) matrix, the ψ^A defined by (8.6.3) are transformed thus:

$$\psi^A \to (\psi^A)' = \pm S^A\ \xi_B \psi^B$$

Indeed, since the right-hand side of (8.6.3) is independent of the choice of the η_C, in (8.6.3), we may put $\eta'_B = Z^A \xi_B \eta_A$, with $Z^A\ \xi_B$ the elements of the inverse matrix S^{-1}, and write

$$\psi^A = \frac{\psi^{BA}\eta_B}{\pm\sqrt{\psi^{CD}\eta_C\eta_D}} = \frac{\psi^{BA}\eta'_B}{\pm\sqrt{\psi^{CD}\eta'_C\eta'_D}}. \tag{8.6.6}$$

Using (8.6.6), for the transformed quantities ψ^A we have

$$(\psi^A)' = \frac{(\psi^{BA})'\,\eta'_B}{\pm\sqrt{(\psi^{CD})'\,\eta'_C\eta'_D}} = \frac{S^A{}_M\psi^{BM}\eta_B}{\pm\sqrt{\psi^{CD}\eta_C\eta_D}} = \pm\, S^A{}_M\,\psi^M.$$

Hence, a geometric object ψ that is transformed via the $S \times S$ group and whose components ψ^{BA} satisfy Eqs. (8.6.2) is equivalent to an object $\psi = \{\pm\psi^A\}$ transformed via the $\{\pm S\}$ group. In particular, it follows from the above stated that in the pseudo-Euclidean space E_4, a second-rank spinor with components ψ^{BA} that satisfy invariant equations (8.6.2) is equivalent to a first-rank spinor with components ψ^A.

If in E_4 the components of a second-rank spinor are not arbitrary but satisfy Eqs. (8.6.2), then, in view of the symmetry properties of γ, the components F, $\overset{*}{F}{}^i$, and $\overset{*}{F}$ defined by (8.5.2) are zeros and we can write (8.5.1) as

$$\psi^{BA} = (1/4)\,[-C^i\gamma_i^{BA} + (1/2)\,C^{ij}\gamma_{ij}^{BA}], \tag{8.6.7}$$

where the vector components C^i and the antisymmetric tensor components C^{ij} are

$$C^i = \gamma^i_{BA}\psi^B\psi^A, \quad C^{ij} = -C^{ij} = \gamma^{ij}_{BA}\psi^B\psi^A. \tag{8.6.8}$$

In view of the definitions (8.6.8), the components C^i and C^{ij} satisfy the following invariant equations:

$$C_{ij}C^{ij} = 0, \quad \varepsilon_{ijks}C^{ij}C^{ks} = 0, \quad C^iC^j + C^i{}_sC^{sj} = 0, \tag{8.6.9}$$

among which there are six independent equations. Tensors with components C^i and C^{ij} that satisfy the algebraic equations (8.6.9) are equivalent to a second-rank spinor with components ψ^{BA} satisfying Eqs. (8.6.2).

Thus, a second-rank spinor with components ψ^{BA} satisfying Eqs. (8.6.2) is equivalent, on the one hand, to a tensor with components C^i and C^{ij} and, on the other, to a first-rank spinor $\boldsymbol{\psi}$. Then, since an equivalence relation is transitive, we have the following.

Theorem [1, 2, 4]. *A first-rank spinor* $\boldsymbol{\psi}$ *in* E_4 *with components* ψ^A *determined to within sign is equivalent to a complex vector and a complex second-rank antisymmetric tensor with components* C^i *and* C^{ij}*, respectively, that satisfy six independent algebraic equations in* (8.6.9). *The one-to-one relationship between the components* C^i *and* C^{ij} *and the components* ψ^A *of the spinor is given by Eqs.* (8.6.3), (8.6.7), *and* (8.6.8).

It is expedient to write Eqs. (8.6.7) and (8.6.8) in matrix notations:

$$\begin{aligned}
&C^i = \psi^T E \gamma^i \psi, \quad C^{ij} = \psi^T E \gamma^{ij} \psi,\\
&\| \psi^{BA} \| = (1/4)\, [C^i \gamma_i - (1/2)\, C^{ij} \gamma_{ij}]\, E^{-1}.
\end{aligned} \tag{8.6.10}$$

Here ψ is a column consisting of four spinor components ψ^A.

If the matrices of the components of the spin-tensor γ_i and E are defined by Eqs. (8.1.7) and (8.1.8), the components C^i and C^{ij} are expressed in terms of the ψ^A thus:

$$\begin{aligned}
C^1 &= 2i\,(\psi^1\psi^3 - \psi^2\psi^4), \quad C^2 = -2\,(\psi^1\psi^3 + \psi^2\psi^4),\\
C^3 &= -2i\,(\psi^1\psi^4 + \psi^2\psi^3), \quad C^4 = 2i\,(-\psi^1\psi^4 + \psi^2\psi^3),\\
C^{14} &= \psi^1\psi^1 - \psi^2\psi^2 + \psi^3\psi^3 - \psi^4\psi^4,\\
C^{24} &= i\,(\psi^1\psi^1 + \psi^2\psi^2 + \psi^3\psi^3 + \psi^4\psi^4),\\
C^{23} &= i\,(\psi^1\psi^1 - \psi^2\psi^2 - \psi^3\psi^3 + \psi^4\psi^4),\\
C^{31} &= -\psi^1\psi^1 - \psi^2\psi^2 + \psi^3\psi^3 + \psi^4\psi^4,\\
C^{34} &= -2\,(\psi^1\psi^2 + \psi^3\psi^4), \quad C^{12} = 2i\,(-\psi^1\psi^2 + \psi^3\psi^4).
\end{aligned} \tag{8.6.11}$$

In this case the expressions for the components ψ^{BA} of a second-rank spinor in terms of the tensor components C^i and C^{ij} are

$$\begin{aligned}
\psi^{11} &= (1/4)\,(-C^{31} - iC^{23} + C^{14} - iC^{24}),\\
\psi^{22} &= (1/4)\,(-C^{31} + iC^{23} - C^{14} - iC^{24}),\\
\psi^{33} &= (1/4)\,(C^{31} + iC^{23} + C^{14} - iC^{24}),\\
\psi^{44} &= (1/4)\,(C^{31} - iC^{23} - C^{14} - iC^{24}),\\
\psi^{14} &= (i/4)\,(C^3 + C^4), \quad \psi^{23} = (i/4)\,(C^3 - C^4),\\
\psi^{31} &= -(1/4)\,(C^2 + iC^1), \quad \psi^{24} = (1/4)\,(-C^2 + iC^1),\\
\psi^{12} &= (1/4)\,(-C^{34} + iC^{12}), \quad \psi^{34} = -(1/4)\,(C^{34} + iC^{12}).
\end{aligned} \tag{8.6.12}$$

8.7 REAL-VALUED TENSORS DETERMINED BY A SPINOR

If the components ψ^{BA} of the second-rank tensor in (8.5.1) are not arbitrary but are expressible in the form of the product $\psi^{BA} = \psi^{+B}\psi^{A}$ with ψ^{A} a first-rank spinor, then we can write (8.5.1) as

$$\psi^{+A}\psi^{B} = (1/4)\ [\Omega e^{AB} - ij^{s}\gamma_{s}^{AB} + (i/2)\ M^{ks}\gamma_{ks}^{AB} - S^{k}\overset{*}{\gamma}{}_{k}^{AB} + N\gamma^{5AB}], \tag{8.7.1}$$

where the components of the tensor $\mathbf{D} = \{\Omega,\ j^{s},\ M^{ks},\ S^{k},\ N\}$ are defined by the following formulas:

$$\begin{aligned}&\Omega = -e_{A\,B}\psi^{+A}\psi^{B},\ j^{s} = -i\gamma^{s}_{AB}\psi^{+A}\psi^{B},\\ &M^{ks} = -i\gamma^{ks}_{AB}\psi^{+A}\psi^{B},\ S^{k} = -\overset{*}{\gamma}{}^{k}_{AB}\psi^{+A}\psi^{B},\\ &N = -\gamma^{5}_{AB}\psi^{+A}\psi^{B}.\end{aligned} \tag{8.7.2}$$

Let us write the definitions (8.7.2) in matrix form:

$$\begin{aligned}&\Omega = \psi^{+}\psi,\ j^{s} = i\psi^{+}\gamma^{s}\psi,\\ &M^{ks} = i\psi^{+}\gamma^{ks}\psi,\ S^{k} = \psi^{+}\overset{*}{\gamma}{}^{k}\psi,\ N = \psi^{+}\gamma^{5}\psi.\end{aligned} \tag{8.7.3}$$

Here ψ^{+} stands for a row of the covariant components ψ^{+}_{A} of the conjugate spinor.

In view of the definitions (8.7.3), the tensor components Ω, j^{s}, M^{ks}, S^{k}, and N are real-valued and satisfy the following bilinear equations [2, 7, 8]:

$$\begin{aligned}&j_{i}j^{i} = -\Omega^{2} - N^{2},\ S_{i}S^{i} = \Omega^{2} + N^{2},\ j_{i}S^{i} = 0,\\ &(1/2)\ M_{ij}M^{ij} = \Omega^{2} - N^{2},\ \varepsilon_{ijks}M^{ij}M^{ks} = 8\Omega N,\\ &\Omega j^{i} = (1/2)\ \varepsilon^{ijks}\ S_{j}M_{ks},\ Nj^{i} = -S_{j}M^{ij},\\ &\Omega S^{i} = (1/2)\ \varepsilon^{ijks}j_{j}M_{ks},\ NS^{i} = -j_{n}M^{in},\\ &j^{i}j^{j} = S^{i}S^{j} + M^{i}{}_{s}M^{js} - \Omega^{2}g^{ij},\\ &\Omega M_{ij} + (1/2)\ N\varepsilon_{ijks}M^{ks} = -\varepsilon_{ijks}j^{k}S^{s}.\end{aligned} \tag{8.7.4}$$

Tensors with components Ω, j^{s}, M^{ks}, S^{k}, and N define a spinor ψ to within a phase factor exp $(i\varphi)$ in all of its components ψ^{A}, where φ is arbitrary and real. The formulas that express the spinor components ψ^{A} in terms of real-valued tensor components Ω, j^{s}, M^{ks}, S^{k}, and N have the following form [2]:

$$\begin{aligned}&\psi^{A} = \psi^{\dot{B}A}\dot{\eta}_{B}/\sqrt{\psi^{\dot{C}D}\dot{\eta}_{C}\eta_{D}},\\ &\|\psi^{\dot{B}A}\| = (1/4)\ [\Omega I - ij^{s}\gamma_{s} + (i/2)\ M^{ks}\gamma_{ks} - S^{k}\overset{*}{\gamma}_{k} + N\gamma^{5}]\ \beta^{-1},\end{aligned} \tag{8.7.5}$$

where η_B are arbitrary and, in general, complex numbers such that $\psi^{\dot{C}D}\dot{\eta}_C\eta_D \neq 0$. If we put $\eta_C = \delta_{BC} \exp(i\varphi)$, where φ is arbitrary and real, then for ψ^A we can write

$$\psi^A = \left(\psi^{\dot{B}A}/\sqrt{\psi^{\dot{B}B}}\right) \exp(i\varphi). \tag{8.7.6}$$

Since the spinor ψ is fully determined by the complex tensor components C^i and C^{ij}, it is obvious that tensors with components Ω, j^s, M^{ks}, S^k, and N must be fully determined in terms of the C^i and C^{ij}. The respective equations are (see [2, 4])

$$\begin{aligned}
4\Omega^2 &= \dot{C}_i C^i - (1/2)\, \dot{C}_{ij} C^{ij}, \quad 4\Omega j^i = i\,(-\dot{C}^{si} C_s + \dot{C}_s C^{si}),\\
4\Omega M^{ij} &= i\,(\dot{C}^i C^j - \dot{C}^j C^i - \dot{C}^{ni} C_n{}^j + \dot{C}^{nj} C_n{}^i),\\
4\Omega S^i &= -(1/2)\, \varepsilon^{ijks} (\dot{C}_j C_{ks} + C_j \dot{C}_{ks}),\\
4\Omega N &= -(1/4)\, \varepsilon_{ijks} \dot{C}^{ij} C^{ks},\\
4Nj^i &= (i/2)\, \varepsilon^{ijks} (-\dot{C}_s C_{jk} + C_s \dot{C}_{jk}),\\
2\varepsilon^{ijks} N M_{ks} &= i\,(\dot{C}^i C^j - \dot{C}^j C^i + \dot{C}^{ni} C_n{}^j - \dot{C}^{nj} C_n{}^i),\\
4NS^i &= \dot{C}_j C^{ij} + C_j \dot{C}^{ij}, \quad 4N^2 = \dot{C}_i C^i + (1/2)\, \dot{C}_{ij} C^{ij},\\
4j^i j^s &= \dot{C}^i C^s + C^i \dot{C}^s + \dot{C}^{ij} C^s{}_j + C^{ij} \dot{C}^s{}_j\\
&+ g^{si}\,[-\dot{C}_j C^j - (1/2)\, \dot{C}_{jn} C^{jn}].
\end{aligned} \tag{8.7.7}$$

Here the tensor complex conjugate components $\dot{C}^i$ and $\dot{C}^{ij}$ are expressible in terms of the components of the conjugate spinor:

$$\dot{C}^i = -\gamma^i_{AB}\psi^{+B}\psi^{+A}, \quad \dot{C}^{ij} = -\gamma^{ij}_{BA}\psi^{+B}\psi^{+A}.$$

The components C^i and C^{ij} of the complex-valued tensors and the components Ω, j^s, M^{ij}, S^k, and N of real-valued tensors are related to each other by the following crossed equations:

$$\begin{aligned}
&C_i j^i = 0, \quad C_i S^i = 0, \quad C_{ij} M^{ij} = 0, \quad \varepsilon_{ijks} C^{ij} M^{ks} = 0,\\
&\Omega C^i = (1/2)\, \varepsilon^{ijks} S_j C_{ks} = -iC_j M^{ji} = iC^{ji} j_j,\\
&NC^i = -(i/2)\, \varepsilon^{ijks} j_j C_{ks} = (i/2)\, \varepsilon^{ijks} C_j M_{ks} = -C^{ij} S_j.
\end{aligned} \tag{8.7.8}$$

If the matrices built on the components of the spin-tensors E and γ_i are given by (8.1.7) and (8.1.8), then the components of a real-valued tensor **D** are expressed in terms of the spinor components

ψ^A and the complex conjugate components $\dot{\psi}^A$ thus:

$$
\begin{aligned}
\Omega &= \dot{\psi}^1\psi^3 + \dot{\psi}^2\psi^4 + \dot{\psi}^3\psi^1 + \dot{\psi}^4\psi^2,\\
j^1 &= \dot{\psi}^1\psi^2 + \dot{\psi}^2\psi^1 - \dot{\psi}^3\psi^4 - \dot{\psi}^4\psi^3,\\
j^2 &= i\,(-\dot{\psi}^1\psi^2 + \dot{\psi}^2\psi^1 + \dot{\psi}^3\psi^4 - \dot{\psi}^4\psi^3),\\
j^3 &= \dot{\psi}^1\psi^1 - \dot{\psi}^2\psi^2 - \dot{\psi}^3\psi^3 + \dot{\psi}^4\psi^4,\\
j^4 &= \dot{\psi}^1\psi^1 + \dot{\psi}^2\psi^2 + \dot{\psi}^3\psi^3 + \dot{\psi}^4\psi^4,\\
M^{23} &= -\dot{\psi}^1\psi^4 - \dot{\psi}^2\psi^3 - \dot{\psi}^3\psi^2 - \dot{\psi}^4\psi^1,\\
M^{31} &= i\,(\dot{\psi}^1\psi^4 - \dot{\psi}^2\psi^3 + \dot{\psi}^3\psi^2 - \dot{\psi}^4\psi^1),\\
M^{12} &= -\dot{\psi}^1\psi^3 + \dot{\psi}^2\psi^4 - \dot{\psi}^3\psi^1 + \dot{\psi}^4\psi^2,\\
M^{14} &= i\,(-\dot{\psi}^1\psi^4 - \dot{\psi}^2\psi^3 + \dot{\psi}^3\psi^2 + \dot{\psi}^4\psi^1),\\
M^{24} &= -\dot{\psi}^1\psi^4 + \dot{\psi}^2\psi^3 + \dot{\psi}^3\psi^2 - \dot{\psi}^4\psi^1,\\
M^{34} &= i\,(-\dot{\psi}^1\psi^3 + \dot{\psi}^2\psi^4 + \dot{\psi}^3\psi^1 - \dot{\psi}^4\psi^2),\\
S^1 &= -\dot{\psi}^1\psi^2 - \dot{\psi}^2\psi^1 - \dot{\psi}^3\psi^4 - \dot{\psi}^4\psi^3,\\
S^2 &= i\,(\dot{\psi}^1\psi^2 - \dot{\psi}^2\psi^1 + \dot{\psi}^3\psi^4 - \dot{\psi}^4\psi^3),\\
S^3 &= -\dot{\psi}^1\psi^1 + \dot{\psi}^2\psi^2 - \dot{\psi}^3\psi^3 + \dot{\psi}^4\psi^4,\\
S^4 &= -\dot{\psi}^1\psi^1 - \dot{\psi}^2\psi^2 + \dot{\psi}^3\psi^3 + \dot{\psi}^4\psi^4,\\
N &= i\,(\dot{\psi}^1\psi^3 + \dot{\psi}^2\psi^4 - \dot{\psi}^3\psi^1 - \dot{\psi}^4\psi^2).
\end{aligned}
\tag{8.7.9}
$$

8.8 ROTATIONS IN FOUR-DIMENSIONAL SPACE AND SPINORS

Suppose that ψ is a four-component spinor in the four-dimensional space E_4 in which an orthonormal basis Э_i is chosen. Consider vectors with components j^i, S^i, p^i, and q^i determined via a spinor:

$$
\begin{aligned}
&j^i = -\,i\gamma^i_{AB}\psi^{+A}\psi^B, \qquad S^i = -\overset{*}{\gamma}{}^i_{AB}\psi^{+A}\psi^B,\\
&q^i = \operatorname{Re} C^i = (1/2)\,\gamma^i_{AB}\,(\psi^A\psi^B - \psi^{+A}\psi^{+B}),\\
&p^i = \operatorname{Im} C^i = -(i/2)\,\gamma^i_{AB}(\psi^A\psi^B + \psi^{+A}\psi^{+B}),
\end{aligned}
\tag{8.8.1}
$$

with ψ^A and ψ^{+A} contravariant components of spinor ψ and conjugate spinor ψ^+ calculated in the Э_i basis. Equations (8.7.4), (8.7.7), and (8.7.8) imply that vectors with components j^i, S^i, p^i, and q^i are mutually orthogonal and have equal moduli:

$$-j_i j^i = S_i S^i = p_i p^i = q_i q^i = \Omega^2 + N^2, \tag{8.8.2}$$
$$j_i S^i = j_i p^i = j_i q^i = S_i p^i = S_i q^i = p_i q^i = 0.$$

If $\rho^2 = \Omega^2 + N^2 \neq 0$, then in E_4 we can introduce an orthonormal basis $\overset{*}{\mathbf{e}}_a$:

$$\overset{*}{\mathbf{e}}_1 = \frac{1}{\rho} p^i \text{Э}_i, \quad \overset{*}{\mathbf{e}}_2 = \frac{1}{\rho} q^i \text{Э}_i, \quad \overset{*}{\mathbf{e}}_3 = \frac{1}{\rho} S^i \text{Э}_i, \quad \overset{*}{\mathbf{e}}_4 = \frac{1}{\rho} j^i \text{Э}_i. \tag{8.8.3}$$

Therefore, if in E_4 we define a spinor with a nonzero invariant ρ, this also defines a pseudo-orthogonal transformation connected with a pair of orthonormal bases Э_i and $\overset{*}{\mathbf{e}}_a$.

From definitions (8.8.1) and (8.8.3) it follows that both a spinor with components ψ and a spinor with components

$$\eta = \alpha\psi + \mu\gamma^5\psi, \tag{8.8.4}$$

where α and μ are arbitrary real numbers that are not simultaneously zero, have corresponding to them one and the same orthonormal basis $\overset{*}{\mathbf{e}}_\alpha$.

Equations (8.7.4), (8.7.7), and (8.7.8) imply that the real-valued tensors **D** determined by the spiner ψ via formula (8.7.2) in the $\overset{*}{\mathbf{e}}_\alpha$ basis are given by the components

$$M^{ij} = \begin{Vmatrix} 0 & \Omega & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & -N & 0 \end{Vmatrix}, \quad \begin{aligned} j^i &= (0,\ 0,\ 0,\ \sqrt{\Omega^2 + N_2}), \\ S^i &= (0,\ 0,\ \sqrt{\Omega^2 + N^2},\ 0), \end{aligned} \tag{8.8.5}$$

For the components of the complex-valued tensors **C** determined by the spinor ψ via formula (8.6.8), in the $\overset{*}{\mathbf{e}}_a$ basis we have

$$C^{ij} = \begin{Vmatrix} 0 & 0 & -iN & \Omega \\ 0 & 0 & -N & -i\Omega \\ iN & N & 0 & 0 \\ -\Omega & i\Omega & 0 & 0 \end{Vmatrix}, \quad C^i = (i\sqrt{\Omega^2 + N^2},\ \sqrt{\Omega^2 + N^2},\ 0,\ 0). \tag{8.8.6}$$

If the matrices built on the components of the spin-tensors E and γ^i are given by (8.1.7) and (8.1.8), then the components of spinor ψ in the $\overset{*}{\mathbf{e}}_a$ basis are

$$\psi^1 = \psi^3 = 0, \quad \psi^2 = i\sqrt{\rho/2}\exp(i\eta/2), \quad \psi^4 = i\sqrt{\rho/2}\exp(-i\eta/2). \tag{8.8.7}$$

Here η is related to Ω and N in the following manner: $\Omega + iN = \rho \exp(i\eta)$.

Let us now consider a point real-valued four-dimensional pseudo-Euclidean space with the sign convention $+ + + -$ (the Minkowski space) with a Cartesian system of coordinates x^i and a covariant basis Э_i. Suppose that $\psi = \psi(x^i)$ is the field of a four-component spinor in the Minkowski space, and this field has corresponding to it fields of real-valued tensors $\mathbf{D}(x^i)$ and fields of complex-valued tensors $\mathbf{C}(x^i)$. At all points of the Minkowski space where the invariant $\rho^2 = \Omega^2 + N^2$ of ψ is nonzero we can introduce a local orthonormal basis $\overset{*}{\mathbf{e}}_a(x^i)$ defined by (8.8.3). Since the vectors of any orthonormal basis of E_4 may be expressed by formulas of the type (8.8.3) if we choose appropriately ψ, it is clear that any, in general nonholonomic, set of orthonormal basis vectors in the Minkowski space can be defined by fixing the four-component spinor field $\psi(x^i)$.

We can show (see [1, 2]) that the Ricci rotation coefficients $\Delta_{k,ij}$, which give the change in the vectors $\overset{*}{\mathbf{e}}_a$ under transformations from one point in the Minkowski space to another, i.e.

$$h_i{}^a d\overset{*}{\mathbf{e}}_a = \Delta_{k,ij} \text{Э}^j \, dx^k, \quad \text{Э}_i = h_i{}^a \overset{*}{\mathbf{e}}_a, \tag{8.8.8}$$

are expressed in terms of the components of spinor ψ in the following manner (here $\partial_k = \partial/\partial x^k$ is the symbol of a partial derivative with respect to variable x^k):

$$\begin{aligned}\Delta_{k,ij} = {} & [1/(\Omega^2 + N^2)]\, [\Omega\, (\psi^+\gamma_{ij}\partial_k\psi - \partial_k\psi^+ \cdot \gamma_{ij}\psi) \\ & + (1/2)\, N\varepsilon_{ijpq}\, (\psi^+\gamma^{pq}\partial_k\psi - \partial_k\psi^+ \cdot \gamma^{pq}\psi)].\end{aligned} \tag{8.8.9}$$

We can now express the Ricci rotation coefficients (8.8.9) in terms of the vector components p^i, q^i, S^i, and j^i:

$$\begin{aligned}\Delta_{k,ij} = {} & \frac{1}{2(\Omega^2 + N^2)} (-j_i\partial_k j_j + j_j\partial_k j_i + S_i\partial_k S_j - S_j\partial_k S_i \\ & + p_i\partial_k p_j - p_j\partial_k p_i + q_i\partial_k q_j - q_j\partial_k q_i).\end{aligned} \tag{8.8.10}$$

This formula enables us to write relationships that can easily be verified (taking into account Eqs. (8.7.4), (8.7.7), and (8.7.8)):

$$\begin{aligned}&\partial_k (p_i/\rho) = -\Delta_{k,ij}\, (p^j/\rho), \quad \partial_k (q_i/\rho) = -\Delta_{k,ij}\, (q^j/\rho), \\ &\partial_k (S_i/\rho) = -\Delta_{k,ij}\, (S^j/\rho), \quad \partial_k (j_i/\rho) = -\Delta_{k,ij}\, (j^j/\rho).\end{aligned} \tag{8.8.11}$$

Thus, by defining in the Minkowski space a four-component spinor ψ we also define a system of orthonormal bases by formulas (8.8.1) and (8.8.3), with the Ricci rotation coefficients for such a system of bases given by (8.8.9).

8.9 INVARIANT SPINOR SUBSPACES

Consider an equation that is linear in the spinor components ψ^A and the components ψ^{+A} of the conjugate spinor and invariant under proper Lorentz transformations of the orthonormal bases $Э_i$ of E_4,

$$\psi^A = i\eta\gamma^{5A}{}_B\psi^B + \mu\psi^{+A}\exp(i\theta), \tag{8.9.1}$$

where η, μ, and θ are real numbers such that $\eta^2 + \mu^2 = 1$. We can easily verify that if χ^A are the components of an arbitrary spinor in E_4, then a spinor with components

$$\psi^A = \chi^A + i\eta\gamma^{5A}{}_B\chi^B + \mu\chi^{+A}\exp(i\theta)$$

satisfies Eq. (8.9.1), provided that the coefficients η and μ satisfy the condition $\eta^2 + \mu^2 = 1$. Hence, Eq. (8.9.1) is solvable with respect to ψ.

Since Eq. (8.9.1) is invariant under transformations of the proper Lorentz group, it defines in the spinor space a certain subspace invariant with respect to this group.

In the spin-basis $\overset{*}{\varepsilon}_A$, where matrices built on the components of the spin-tensors E and γ_i are defined in (8.1.7) and (8.1.8), Eqs. (8.9.1) can be written thus:

$$(1-\eta)\psi^1 = -\mu\dot{\psi}^4\exp(i\theta),\quad (1-\eta)\psi^2 = \mu\dot{\psi}^3\exp(i\theta),$$

$$(1+\eta)\psi^3 = \mu\dot{\psi}^2\exp(i\theta),\quad (1+\eta)\psi^4 = -\mu\dot{\psi}^1\exp(i\theta). \tag{8.9.2}$$

The complex tensors **C** and real tensors **D** corresponding to spinors from the subspace (8.9.1) have a special form, which is determined by the following restricting conditions:

$$\begin{aligned} &S^i = -\eta j^i,\quad C^j = i\mu j^j e^{i\theta},\quad \Omega = N = 0,\\ &C^{jm} = (i/2)\,\eta\varepsilon^{jmks}C_{ks} + i\mu e^{i\theta}M^{jm},\\ &M^{jm} = -\,i\mu e^{-i\theta}C^{jm} - (i/2)\,\eta\varepsilon^{jmks}M_{ks}. \end{aligned} \tag{8.9.3}$$

At $\eta = \theta = 0$ and $\mu = 1$, the subspace (8.9.1) is given by the equation $\psi^{+A} = \psi^A$. The spinor with components ψ^A satisfying this equation is called a real-valued spinor. Depending on the class of the spinor representation we are considering, the subspace of real-valued spinors may be invariant under transformations of either the proper or the full Lorentz group.

For real-valued spinors, Eqs. (8.9.3) transform into

$$\Omega = N = S^i = 0,\ C^m = ij^m,\ C^{mn} = iM^{mn}. \tag{8.9.4}$$

Hence, the tensors **C** and **D** defined by real-valued spinors have the special form

$$\mathbf{D} = \{0,\ j^i,\ M^{ij},\ 0,\ 0\},\quad \mathbf{C} = \{ij^m,\ iM^{mn}\}.$$

The algebraic equations (8.7.4), (8.7.7), and (8.7.8) connecting the components of **C** and **D** for real-valued spinors become

$$\begin{gathered} j_i j^i = 0, \ M_{ij}M^{ij} = 0, \ \varepsilon_{ijks}M^{ij}M^{ks} = 0, \\ j_i M^{ij} = 0, \ \varepsilon^{ijks} j_s M_{jk} = 0, \ j^i j^n = M^i{}_j M^{nj}. \end{gathered} \tag{8.9.5}$$

At $\mu = 0$ and $\eta = \pm 1$, Eqs. (8.9.1) can be written thus:

$$(I - i\gamma^5)\psi = 0 \text{ or } (I + i\gamma^5)\psi = 0; \tag{8.9.6}$$

they define the subspace of so-called half-spinors. We can easily see that if the matrices γ_i are given by (8.1.7), the first equation in (8.9.6) yields $\psi^3 = \psi^4 = 0$ while the second yields $\psi^1 = \psi^2 = 0$. Hence, half-spinors are defined by no more than two independent complex parameters.

For the tensors **C** and **D** defined by half-spinors we have

$$\begin{gathered} \Omega = N = M^{ij} = 0, \ j^i = \mp S^i, \\ C^i = 0, \quad C^{ij} = \pm (i/2)\, \varepsilon^{ijks} C_{ks}. \end{gathered} \tag{8.9.7}$$

Here the upper sign corresponds to the first equation in (8.9.6) while the lower to the second.

The algebraic equations that link the tensors **C** and **D** defined by half-spinors are

$$j_i j^i = 0, \ j_i C^{ij} = 0, \ 4j^i j^j = C^i{}_s \dot{C}^{js} + \dot{C}^i{}_s C^{js}.$$

If we study the compatibility of equations of a general type, linear in ψ and ψ^+ and invariant under proper Lorentz transformations,

$$\alpha\psi^A + \eta\gamma^{5A}{}_B\psi^B + \mu\psi^{+A} + \varkappa\gamma^{5A}{}_B\psi^{+B} = 0, \tag{8.9.8}$$

we can show that the spinor subspace defined by Eq. (8.9.1) is the most general subspace of type (8.9.8) that is invariant under proper Lorentz transformations.

8.10 SPINORS IN THREE-DIMENSIONAL EUCLIDEAN SPACE

Let us take the three-dimensional real-valued Euclidean space E_3 with an orthonormal basis $Э_\alpha$, $\alpha = 1, 2, 3$. Suppose that SO_3 is the group of proper orthogonal transformations of the $Э_\alpha$ basis:

$$Э'_\alpha = l^\beta{}_\alpha Э_\beta, \ \det \| l^\beta{}_\alpha \| = 1. \tag{8.10.1}$$

A spinor representation of the orthogonal group SO_3 is given by the group of matrix pairs $\{\pm S\}$ defined by the equations

$$ {}^{\beta}{}_{\alpha}\sigma_\beta = S^{-1}\sigma_\alpha S, \ S^{\mathrm{T}} E\, S = E; \tag{8.10.2}$$

where σ_α are the Pauli matrices

$$\sigma_1 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad \sigma_2 = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}, \quad \sigma_3 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}, \tag{8.10.3}$$

and the 2×2 matrix E is defined in the following manner:

$$E = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{8.10.4}$$

A first-rank spinor in E_3 is a geometric object of the type $\boldsymbol{\psi} = \pm\psi^A\varepsilon_A$, where the pairs of contravariant components $\pm\psi^A$ and the spin-bases $\pm\varepsilon_A$ are referred to an orthonormal basis $\mathfrak{Э}_\alpha$ of E_3 and under an orthogonal transformation (8.10.1) of this basis are transformed thus:

$$\pm\psi'^A = \pm S^A{}_B\psi^B, \qquad \pm\varepsilon'_A = \pm Z^B{}_A\varepsilon_B, \tag{8.10.5}$$

with $S^A{}_B$ the elements of matrix S given by Eqs. (8.10.2), and $Z^B{}_A$ the elements of the inverse S^{-1}.

The covariant components ψ_B of the same spinor are obtained by using the metric spinor E (whose components are defined in (8.10.4)) with elements e_{BA}:

$$\psi_B = e_{BA}\psi^A, \tag{8.10.6}$$

or $\psi_1 = \psi^2$ and $\psi_2 = -\psi^1$.

The second equation in (8.10.2) implies that the components of the metric spinor E are invariant under transformations of basis $\mathfrak{Э}_\alpha$, while the first implies that the Pauli matrices σ_α form the components of invariant spin-tensors with the following index structure: $\sigma_\alpha = \| \sigma^B{}_{\alpha A} \|$.

In E_3 the raising and lowering spinor indices is done by contracting the spinor with the components e_{BA} and e^{BA} of the metric spinor (always with respect to the second index).

A spinor given by its covariant components $\psi^+_A = \dot{\psi}^A$ is said to be conjugate to the spinor with components ψ^A (in one and the same basis). We denote the row of covariant components of the conjugate spinor by $\boldsymbol{\psi}^+ = \dot{\boldsymbol{\psi}}^{\mathrm{T}} = (\psi^+_1, \psi^+_2)$. The contravariant components ψ^{+B} of the conjugate spinor are obtained by using the metric spinor:

$$\psi^{+B} = e^{BA}\,\psi^+_A. \tag{8.10.7}$$

Employing (8.10.4) and (8.10.7), we find that $\psi^{+1} = -\dot{\psi}^2$ and $\psi^{+2} = \dot{\psi}^1$.

8.11 TENSOR REPRESENTATION OF SPINORS IN THREE-DIMENSIONAL EUCLIDEAN SPACE

In view of the definitions (8.10.3) and (8.10.4), the components of the spin-tensors E and σ_α satisfy the following identities:

$$\begin{aligned}
&2e_{DE}e_{BA} = -e_{DA}e_{EB} + \sigma^{\alpha}_{DA}\sigma_{\alpha EB},\\
&2\sigma^{\alpha}_{DE}e_{BA} = e_{DA}\sigma^{\alpha}_{EB} - \sigma^{\alpha}_{DA}e_{EB} - i\varepsilon^{\alpha}{}_{\beta\eta}\sigma^{\beta}_{DA}\sigma^{\eta}_{EB},\\
&2e_{DE}\sigma^{\alpha}_{BA} = e_{DA}\sigma^{\alpha}_{EB} - \sigma^{\alpha}_{DA}e_{EB} + i\varepsilon^{\alpha}{}_{\beta\eta}\sigma^{\beta}_{DA}\sigma^{\eta}_{EB}, \qquad (8.11.1)\\
&2\sigma^{\alpha}_{DE}\sigma^{\beta}_{BA} = -\delta^{\alpha\beta}(e_{DA}e_{EB} + \sigma^{\eta}_{DA}\sigma_{\eta EB}) + \sigma^{\alpha}_{DA}\sigma^{\beta}_{EB}\\
&\quad + \sigma^{\beta}_{DA}\sigma^{\alpha}_{EB} - i\varepsilon^{\alpha\beta\eta}(e_{DA}\sigma_{\eta EB} + \sigma_{\eta DA}e_{EB}).
\end{aligned}$$

The definitions (8.10.3) and (8.10.4) show that the four matrices E^{-1} and $\sigma_\alpha E^{-1} = \| \sigma^{BA}_{\alpha} \|$ are linearly independent and, hence, form a basis in the space of 2×2 matrices. Each 2-by-2 matrix $\| \psi^{BA} \|$ can be represented in the form

$$\psi^{BA} = (1/2)(Fe^{BA} + F^{\alpha}\sigma^{BA}_{\alpha}), \qquad (8.11.2)$$

where

$$F = -e_{BA}\psi^{BA}, \quad F^{\alpha} = -\sigma^{\alpha}_{BA}\psi^{BA}. \qquad (8.11.3)$$

Obviously, if ψ^{BA} are the components of a second-rank spinor, F is an invariant and F^{α} are components of a vector. If ψ^{BA} in (8.11.2) is written in the form of a product of first-rank spinor components, $\psi^{BA} = \psi^{B}\psi^{A}$, then, in view of the antisymmetry of the components e_{BA} of the metric spinor, the invariant F is zero, and we can write formula (8.11.2) for this case as

$$\psi^{BA} = -(1/2)\, C^{\alpha}\sigma^{BA}_{\alpha}, \qquad (8.11.4)$$

where for the complex components C^{α} we have

$$C^{\alpha} = \sigma^{\alpha}_{BA}\psi^{B}\psi^{A}, \qquad (8.11.5)$$

or

$$C^{1} = \psi^{1}\psi^{1} - \psi^{2}\psi^{2}, \quad C^{2} = i(\psi^{1}\psi^{1} + \psi^{2}\psi^{2}), \quad C^{3} = -2\psi^{1}\psi^{2}.$$

Contracting the first identity in (8.11.1) with the components of the spinor $\psi^{D}\psi^{E}\psi^{B}\psi^{A}$, we find that the vector with components C_{α} is isotropic:

$$C_{\alpha}C^{\alpha} = 0. \qquad (8.11.6)$$

In a way similar to the proof for spinors in four-dimensional space, we can show that an isotropic vector with components C^{α} defines a spinor ψ according to the formulas

$$\psi^A = \psi^{BA}\eta_B / \left(\pm\sqrt{\psi^{CD}\eta_C\eta_D}\right), \quad \psi^{BA} = -(1/2)\, C^\alpha \sigma_\alpha^{BA}, \tag{8.11.7}$$

where η_B ($B = 1, 2$) are, in general, arbitrary complex numbers such that $\psi^{CD}\eta_C\eta_D \neq 0$. If the components C^α of a vector are related to each other by Eq. (8.11.6), the definition (8.11.7) for ψ^A does not depend on the choice of η_B.

Thus, a first-rank spinor ψ in the three-dimensional space E_3 is equivalent to a complex isotropic vector $\mathbf{C} = C^\alpha Э_\alpha$. The one-to-one relationship between the two is given by (8.11.5) and (8.11.7).

We write the second formula in (8.11.7) component-wise:

$$\psi^{11} = (1/2)\,(C^1 - iC^2),\ \psi^{22} = -(1/2)\,(C^1 + iC^2),$$
$$\psi^{12} = \psi^{21} = -(1/2)\, C^3.$$

If the components ψ^{BA} of the second-rank spinor in (8.11.2) are expressible in terms of a product, $\psi^{BA} = \psi^{+B}\psi^A$, of the components ψ^A of a first-rank spinor and the components ψ^{+B} of the conjugate spinor, we can write (8.11.2) as

$$\psi^{+B}\psi^A = (1/2)\,(\rho e^{BA} + j^\alpha \sigma_\alpha^{BA}), \tag{8.11.8}$$

where

$$\rho = -e_{BA}\psi^{+B}\psi^A, \quad j^\alpha = -\sigma_{BA}^\alpha \psi^{+B}\psi^A, \tag{8.11.9}$$

or

$$\rho = \dot\psi^1\psi^1 + \dot\psi^2\psi^2,\ j^1 = \dot\psi^1\psi^2 + \dot\psi^2\psi^1,$$
$$j^2 = i\,(\dot\psi^2\psi^1 - \dot\psi^1\psi^2),\ j^3 = \dot\psi^1\psi^1 - \dot\psi^2\psi^2. \tag{8.11.10}$$

The invariant ρ and the vector components j^α are real-valued, with $\rho \geqslant 0$. Contracting the first identity in (8.11.1) with the components of the spinor $\psi^{+D}\psi^E\psi^{+B}\psi^A$, we find that the components ρ and j^α satisfy the equation

$$j_\alpha j^\alpha = \rho^2. \tag{8.11.11}$$

The invariant ρ and the vector components j^α satisfying Eq. (8.11.11) define a spinor to within a phase factor of each of its components equal to $\exp(i\varphi)$, where φ is arbitrary and real.

Contracting the identities (8.11.1) with the components of the spinor $\psi^{+D}\psi^E\psi^B\psi^{+A}$, we find ρ and the j^α expressed in terms of the vector components C^α:

$$\begin{aligned} 2\rho^2 &= \dot C_\alpha C^\alpha, \quad 2\rho j^\alpha = -i\varepsilon^{\alpha\beta\eta}\dot C_\beta C_\eta, \\ 2j^\alpha j^\beta &= \delta^{\alpha\beta}\dot C_\eta C^\eta - \dot C^\alpha C^\beta - \dot C^\beta C^\alpha. \end{aligned} \tag{8.11.12}$$

Contracting the identities (8.11.1) with the components of the spinor $\psi^{+D}\psi^{E}\psi^{B}\psi^{A}$, we also find that ρ, j^α, and C^α are related by crossed equations of the type

$$C_\alpha j^\alpha = 0,\ \rho C^\alpha = i\varepsilon^{\alpha\beta\eta} j_\beta C_\eta,\ j^\alpha C^\beta - j^\beta C^\alpha = -i\varepsilon^{\alpha\beta\eta}\rho C_\eta. \qquad (8.11.13)$$

8.12 ROTATIONS IN THREE-DIMENSIONAL SPACE AND SPINORS

Let us write the complex vector components C^α in the form $C^\alpha = p^\alpha + iq^\alpha$, where p^α and q^α define two real-valued vectors:

$$p^\alpha = \operatorname{Re} C^\alpha = (1/2)\,(C^\alpha + \dot{C}^\alpha),\ q^\alpha = \operatorname{Im} C^\alpha$$
$$= (i/2)\,(\dot{C}^\alpha - C^\alpha). \qquad (8.12.1)$$

Equations (8.11.6), (8.11.11)-(8.11.13) imply that the three vectors with components p^α, q^α, and j^α are mutually orthogonal and have equal moduli:

$$j_\alpha p^\alpha = j_\alpha q^\alpha = p_\alpha q^\alpha = 0,\ j_\alpha j^\alpha = p_\alpha p^\alpha = q_\alpha q^\alpha = \rho^2. \qquad (8.12.2)$$

Hence, for each nonzero spinor ψ the vectors with components p^α, q^α, and j^α form an orthonormal basis. For this reason, defining a first-rank nonzero spinor in E_3 determines a rotation, which is described by going over from basis Э_α to basis $\overset{*}{\mathbf{e}}_a$,

$$\overset{*}{\mathbf{e}}_1 = (p^\alpha/\rho)\,\text{Э}_\alpha,\ \overset{*}{\mathbf{e}}_2 = (q^\alpha/\rho)\,\text{Э}_\alpha,\ \overset{*}{\mathbf{e}}_3 = (j^\alpha/\rho)\,\text{Э}_\alpha, \qquad (8.12.3)$$

and an isotropic stretching $\text{Э}'_\alpha = \rho\text{Э}_\alpha$.

The scale factors $h_\alpha{}^a$ and $h^\alpha{}_a$ that connect Э_α and $\overset{*}{\mathbf{e}}_a$, i.e.

$$\overset{*}{\mathbf{e}}_a = h^\alpha{}_a\text{Э}_\alpha,\ \text{Э}_\alpha = h_\alpha{}^a\overset{*}{\mathbf{e}}_a, \qquad (8.12.4)$$

are defined, according to (12.3), thus:

$$\| h^\alpha{}_a \| = \| h_\alpha{}^a \| = \begin{Vmatrix} p^1 & q^1 & j^1 \\ p^2 & q^2 & j^2 \\ p^3 & q^3 & j^3 \end{Vmatrix}. \qquad (8.12.5)$$

One can easily see that both spinor ψ and spinor $\eta\psi$, where η is an arbitrary real nonzero number, correspond to the same basis $\overset{*}{\mathbf{e}}_a$ and, hence, the same scale factor $h^\alpha{}_a$ and $h_\alpha{}^a$.

Let us now suppose that the components of spinor ψ are functions of a scalar parameter t. In the Э_α basis of E_3 we take a vector with components

$$\Omega^\alpha = (i/\rho)\,[\psi^+\sigma^\alpha\, d\psi/dt - (d\psi^+/dt)\,\sigma^\alpha\psi]$$

$$= (i/\rho)\,\sigma^\alpha_{BA}\,(-\psi^{+B}\, d\psi^A/dt + \psi^A d\psi^{+B}/dt), \tag{8.12.6}$$

where the invariant ρ is given by (8.11.9). We can easily see that, in view of the definition (8.12.6), we have

$$\psi^{+D}\psi^E\Omega^\alpha = \frac{i}{\rho}\,\sigma^\alpha_{BA}\left[-\psi^{+B}\psi^E\frac{d}{dt}(\psi^{+D}\psi^A)\right.$$

$$\left.+\,\psi^E\psi^A\frac{d}{dt}(\psi^{+D}\psi^{+B})\right]. \tag{8.12.7}$$

Contracting (8.12.7) in indices D and E with the components e_{DE} of the metric spinor and allowing for (8.11.1), we can express Ω^α in terms of the components j^α, p^α, and q^α:

$$\Omega^\alpha = \frac{1}{2\rho^2}\,\varepsilon^{\alpha\beta\eta}\left(j_\vartheta\frac{dj_\eta}{dt} + p_\beta\frac{dp_\eta}{dt} + q_\beta\frac{dq_\eta}{dt}\right). \tag{8.12.8}$$

This implies that

$$d\,(j^\alpha/\rho)\,dt = \varepsilon^{\alpha\beta\lambda}\Omega_\beta\,(j_\lambda/\rho),\quad d\,(p^\alpha/\rho)\,dt = \varepsilon^{\alpha\beta\lambda}\Omega_\beta\,(p_\lambda/\rho),$$

$$d\,(q^\alpha/\rho)\,dt = \varepsilon^{\alpha\beta\lambda}\Omega_\beta\,(q_\lambda/\rho), \tag{8.12.9}$$

which are simply identities in view of the definitions for Ω^α, ρ, j^α p^α, and q^α.

If E_3 is the physical three-dimensional space and t the absolute time, Eqs. (8.12.9) imply that the vector with components Ω^α defined by (8.12.6) is the vector of angular velocity of the rotation of the basis $\overset{*}{\mathbf{e}}_a$ defined by spinor $\boldsymbol{\psi}$.

Now in the point three-dimensional Euclidean space G_3 defined on the manifold x^α ($\alpha = 1, 2, 3$) let us consider a spinor field $\boldsymbol{\psi}$ (x^α) given by the components ψ^A (x^α) with respect to a Cartesian system of coordinates with variables x^α. The spinor field $\boldsymbol{\psi}$ (x^α) defines at each point of G_3 an orthonormal basis $\overset{*}{\mathbf{e}}_a$ (x^α), which is determined by $\boldsymbol{\psi}$ via formulas (8.12.3). The set $\{\overset{*}{\mathbf{e}}_a\ (x^\alpha)\}$ can be taken as a system of coordinates (in general nonholonomic) in G_3.

Take the quantities $\Delta_{\lambda,\alpha\beta}$ determined by the spinor field $\boldsymbol{\psi}$ (x^α) and the conjugate spinor field $\boldsymbol{\psi}^+$ (x^α):

$$\Delta_{\lambda,\alpha\beta} = (i/\rho)\,\varepsilon_{\alpha\beta\eta}\sigma^\eta_{BA}\,(-\psi^{+B}\partial_\lambda\psi^A + \psi^A\partial_\lambda\psi^{+B}). \tag{8.12.10}$$

Contracting

$$\psi^{+D}\psi^E\,\Delta_{\lambda,\alpha\beta} = (i/\rho)\,\varepsilon_{\alpha\beta\eta}\sigma^\eta_{BA}\,[-\psi^{+B}\psi^E\partial_\lambda\,(\psi^{+D}\psi^A)$$

$$+\,\psi^E\psi^A\partial_\lambda\,(\psi^{+D}\psi^{+B})],$$

which follow from the definition (8.12.10), with the components e_{DE} of the metric spinor with respect to indices D and E and allowing for (8.11.1), we find that

$$\Delta_{\lambda,\alpha\beta} = (1/2\rho^2)\,(p_\alpha\partial_\lambda p_\beta - p_\beta\partial_\lambda p_\alpha + q_\alpha\partial_\lambda q_\beta - q_\beta\partial_\lambda q_\alpha + j_\alpha\partial_\lambda j_\beta - j_\beta\partial_\lambda j_\alpha). \tag{8.12.11}$$

Using the notations of (8.12.5), we can write the above formula as

$$\Delta_{\lambda,\alpha\beta} = (1/2)\,\delta_{ab}\,(h_\alpha{}^a\partial_\lambda h_\beta{}^b - h_\beta{}^a\partial_\lambda h_\alpha{}^b). \tag{8.12.12}$$

Equations (8.12.12) constitute the definition of the Ricci symbols corresponding to bases $\overset{*}{\mathbf{e}}_a$. These symbols enable us to calculate derivatives of vectors $\overset{*}{\mathbf{e}}_a$:

$$\partial_\lambda\overset{*}{\mathbf{e}}_a = (h^\alpha{}_a h^\beta{}_b\,\Delta_{\lambda,\alpha\beta})\,\overset{*}{\mathbf{e}}{}^b = \Delta_{\lambda,ab}\overset{*}{\mathbf{e}}{}^b. \tag{8.12.13}$$

If in Eqs. (8.12.13) we express the $\overset{*}{\mathbf{e}}_a$ in terms of the $\mathrm{Э}_\alpha$ via (8.12.3) from Eqs. (8.12.13) we arrive at equations that link the derivatives $\partial_\lambda j_\alpha$, $\partial_\lambda p_\alpha$, and $\partial_\lambda q_\alpha$ with the components j_α, p_α, and q_α:

$$\begin{aligned}\partial_\lambda\,(j_\alpha/\rho) &= -\Delta_{\lambda,\alpha}{}^\beta\,(j_\beta/\rho),\\ \partial_\lambda\,(p_\alpha/\rho) &= -\Delta_{\lambda,\alpha}{}^\beta\,(p_\beta/\rho),\\ \partial_\lambda\,(q_\alpha/\rho) &= -\Delta_{\lambda,\alpha}{}^\beta\,(q_\beta/\rho).\end{aligned} \tag{8.12.14}$$

We can also obtain Eqs. (8.12.14) by directly contracting (8.12.11) with the components p^α, q^α, and j^α.

Instead of Ricci symbols $\Delta_{\lambda,\alpha\beta}$ it is often expedient to use their duals $\Delta_{\lambda\alpha}$:

$$\Delta_{\lambda\alpha} = (1/2)\,\varepsilon_\alpha{}^{\mu\nu}\Delta_{\lambda,\mu\nu} = (i/\rho)\,[\psi^+\sigma_\alpha\,\partial_\lambda\psi - (\partial_\lambda\psi^+)\,\sigma_\alpha\psi].$$

8.13 TENSOR REPRESENTATION OF DIFFERENTIAL SPINOR EQUATIONS IN THE MINKOWSKI SPACE

Suppose that $\psi^A = \psi^A\,(x^i)$ is the field of contravariant components of a first-rank spinor in the Minkowski space with a Cartesian system of coordinates with variables x^i, $i = 1, 2, 3, 4$. Let us consider the following differential equations:

$$\gamma^{B}{}^{i}_{A}\partial_i\psi^A + [\varkappa\delta^B_A + i\varkappa_j\gamma^{B}{}^{j}_{A} + (i/2)\,\varkappa_{js}\gamma^{B}{}^{j}_{A}{}^{s} + \overset{*}{\varkappa}_j\overset{*}{\gamma}{}^{B}{}^{j}_{A} + \overset{*}{\varkappa}\gamma^{B}{}^{5}_{A}]\psi^A = 0. \tag{8.13.1}$$

Here $\varkappa$, $\varkappa_j$, $\varkappa_{js}$, $\overset{*}{\varkappa}_j$, and $\overset{*}{\varkappa}$ are real-valued and are fixed functions of the x^i or are given by additional equations.

From Eqs. (8.13.1) follow the equations for the conjugate spinor field:

$$-\gamma^{B\,i}_{\;A}\partial_i\psi^+_B + \psi^+_B\,[\varkappa\delta^B_A + i\varkappa_j\gamma^{B\,j}_{\;A} + (i/2)\,\varkappa_{js}\gamma^{B\,j\;s}_{\;A}$$
$$+ \overset{*}{\varkappa}_j\overset{*}{\gamma}{}^{B\,j}_{\;A} + \overset{*}{\varkappa}\gamma^{B\,5}_{\;A}] = 0. \tag{8.13.2}$$

It is expedient to use these equations in matrix form:

$$\gamma^i\partial_i\psi + [\varkappa I + i\varkappa_j\gamma^j + (i/2)\,\varkappa_{js}\gamma^{js} + \overset{*}{\varkappa}_j\overset{*}{\gamma}{}^j + \overset{*}{\varkappa}\gamma^5]\,\psi = 0,$$
$$-\partial_i\psi^+\gamma^i + \psi^+\,[\varkappa I + i\varkappa_j\gamma^j + (i/2)\,\varkappa_{js}\gamma^{js} \tag{8.13.3}$$
$$+ \overset{*}{\varkappa}_j\overset{*}{\gamma}{}^j + \overset{*}{\varkappa}\gamma^5] = 0.$$

Equations (8.13.1) contain, as particular cases, the well-known spinor equations of quantum field theory (Dirac equations, Heissenberg equations, etc.); for special values of $\varkappa$, $\varkappa_j$, $\varkappa_{js}$, $\overset{*}{\varkappa}_j$, and $\overset{*}{\varkappa}$, Eqs. (8.13.1) are used in the mechanics of magnetizable continuous media [1, 2]. Below we study a closed system of differential tensor equations in components of real-valued tensors determined by a spinor. This system is a consequence of spinor equations (8.13.1) (tensor equations in components of complex-valued tensors C equivalent to spinor equations (8.13.1) are given in [2, 4]).

If we contract Eqs. (8.13.1) with the spin-tensor components ψ^+_B, $\psi^+_C\gamma^{C\,i}_{\;B}$, $\psi^+_C\gamma^{C\,i\;j}_{\;B}$, $\psi^+_C\overset{*}{\gamma}{}^{C\,i}_{\;B}$, and $\psi^+_C\gamma^{C\,5}_{\;B}$ with respect to index B and then, separate the real and imaginary parts, we find that

$$\text{(a)}\quad \partial_i j^i = 0,$$
$$\text{(b)}\quad (i/2)\,e_{AB}\,(\psi^{+A}\partial^i\psi^B - \psi^B\partial^i\psi^{+A}) = (1/2)\,\partial_j M^{ij} + \varkappa_j{}^i$$
$$- \varkappa^i\Omega + (1/2)\,\varepsilon^{ijks}(\overset{*}{\varkappa}_j M_{ks} + \varkappa_{jk}S_s),$$
$$\text{(c)}\quad (1/2)\,(g^{ki}\overset{j}{\gamma}_{AB} - g^{kj}\overset{i}{\gamma}_{AB})\,(\psi^{+A}\partial_k\psi^B - \psi^B\partial_k\psi^{+A})$$
$$= (1/2)\,\varepsilon^{ijks}\partial_k S_s + \varkappa^i j^j - \varkappa^j j^i + \varkappa^i{}_s M^{js} - \varkappa^j{}_s M^{is} + \overset{*}{\varkappa}{}^i S^j - \overset{*}{\varkappa}{}^j S^i,$$
$$\text{(d)}\quad (i/2)\,\overset{5}{\gamma}_{AB}\,(\psi^{+A}\partial^i\psi^B - \psi^B\partial^i\psi^{+A}) = (1/4)\,\varepsilon^{ijks}\partial_j M_{ks} \tag{8.13.4}$$
$$- \varkappa^i N + \varkappa^{ij}S_j + \overset{*}{\varkappa}_j M^{ij} + \overset{*}{\varkappa} j^i,$$
$$\text{(e)}\quad (1/2)\,\partial_i S^i - \varkappa N - (1/4)\,\varepsilon^{ijks}\varkappa_{ij}M_{ks} + \overset{*}{\varkappa}\Omega = 0.$$

Introducing the notation

$$\overset{*}{P}{}_i{}^j = \alpha\,[\psi^+\gamma^j\partial_i\psi - \partial_i\psi^+\gamma^j\psi - \delta^j_i\,(\psi^+\gamma^s\partial_s\psi - \partial_s\psi^+\gamma^s\psi)]$$
$$\equiv -\alpha\,(\overset{j}{\gamma}_{AB}\delta^s_i - \overset{s}{\gamma}_{AB}\delta^j_i)\,(\psi^{+A}\partial_s\psi^B - \psi^B\partial_s\psi^{+A}),\quad \alpha = \text{const},$$

we can write Eq. (8.13.4c) in the following form:

$$\overset{*}{P}{}^{ij}-\overset{*}{P}{}^{ji}+\alpha\varepsilon^{ijks}\partial_k S_s$$
$$+2\alpha\left(\varkappa^i j^j-\varkappa^j j^i+\varkappa^i{}_s M^{js}-\varkappa^j{}_s M^{is}+\overset{*}{\varkappa}{}^i S^j-\overset{*}{\varkappa}{}^j S^i\right)=0. \tag{8.13.5}$$

A direct check shows that, in view of Eqs. (8.13.1), the tensor components $\overset{*}{P}{}_i{}^j$ satisfy the following equation:

$$\partial_j \overset{*}{P}{}_i{}^j = 2\alpha\,[\varkappa\partial_i\Omega + \varkappa_j\partial_i j^j + (1/2)\,\varkappa_{sj}\,\partial_i M^{sj} + \overset{*}{\varkappa}_j\,\partial_i S^j + \overset{*}{\varkappa}\,\partial_i N]. \tag{8.13.6}$$

Let us express the $\overset{*}{P}{}_i{}^j$ in terms of the real-valued tensor components Ω, j^i, M^{ij}, S^i, and N. We can easily see that for any differentiable fields $\psi^A(x^i)$ and $\psi^{+A}(x^i)$ we have the identity

$$\psi^{+D}\psi^E\,(\psi^{+B}\,d\psi^A-\psi^A\,d\psi^{+B})-\psi^{+B}\psi^A\,(\psi^{+D}\,d\psi^E-\psi^E\,d\psi^{+D})$$
$$\equiv\psi^E\psi^{+B}d\,(\psi^{+D}\psi^A)-\psi^{+D}\psi^A d\,(\psi^{+B}\psi^E). \tag{8.13.7}$$

Allowing for this identity and in view of definition (8.7.2) we can write

$$\psi^{+D}\psi^E\overset{*}{P}{}_i{}^j=-\alpha\,(\gamma^j_{AB}\delta^s_i-\gamma^s_{AB}\delta^j_i)\,[\psi^E\psi^{+B}\partial_s(\psi^{+D}\psi^A)$$
$$-\psi^{+D}\psi^A\partial_s(\psi^{+B}\psi^E)]-\alpha i\,(j^j\delta^s_i-j^s\delta^j_i)\,(\psi^{+D}\partial_s\psi^E-\psi^E\partial_s\psi^{+D}). \tag{8.13.8}$$

Contracting Eqs. (8.13.8) with the components e_{DE} of the metric tensor and with the spin-tensor components $\overset{5}{\gamma}_{DE}$ yields

$$\begin{aligned}
\Omega\overset{*}{P}{}_i{}^j&=\alpha\,(\delta^j_i\delta^n_m-\delta^j_m\delta^n_i)\,[ij^m e_{ED}\,(\psi^{+D}\partial_n\psi^E-\psi^E\partial_n\psi^{+D})\\
&+S^m\partial_n N+M^{ms}\partial_n j_s],\\
N\overset{*}{P}{}_i{}^j&=\alpha\,(\delta^j_i\delta^n_m-\delta^j_m\delta^n_i)\,[ij^m\overset{5}{\gamma}_{DE}(\psi^{+D}\partial_n\psi^E\\
&-\psi^E\partial_n\psi^{+D})-S^m\partial_n\Omega+(1/2)\,\varepsilon^{mksq}M_{ks}\partial_n j_q].
\end{aligned} \tag{8.13.9}$$

By means of Eqs. (8.13.4b, d), we can transform Eqs. (8.13.9) into

$$\begin{aligned}
\Omega\overset{*}{P}{}_i{}^j&=\alpha\,(j^j\partial_s M_i{}^s-M^{js}\partial_i j_s-S^j\partial_i N)\\
&+2\alpha\{j^j\,[\varkappa j_i-\Omega\varkappa_i+(1/2)\,\varepsilon_{ikms}\,(\overset{*}{\varkappa}{}^k M^{ms}+\varkappa^{km}S^s)]\\
&+\delta^j_i\Omega\,[\varkappa\Omega+\varkappa_s j^s+(1/2)\,\varkappa_{ks}M^{ks}+\varkappa_m S^m+\overset{*}{\varkappa}N]\},\\
N\overset{*}{P}{}_i{}^j&=\alpha\,[(1/2)\,j^j\varepsilon_{iksm}\partial^k M^{sm}-(1/2)\,\varepsilon^{jksm}M_{ks}\partial_i j_m+S^j\partial_i\Omega]\\
&+2\alpha\,[j^j\,(-N\varkappa_i+\varkappa_{im}S^m-\overset{*}{\varkappa}{}^s M_{is}+\overset{*}{\varkappa}j_i)\\
&+\delta^j_i N\,(\varkappa\Omega+\varkappa_s j^s+(1/2)\,\varkappa_{ks}M^{ks}+\overset{*}{\varkappa}_m S^m+\overset{*}{\varkappa}N)].
\end{aligned} \tag{8.13.10}$$

The system of equations (8.13.4a, c, e) and (8.13.6), in which the tensor components $\overset{*}{P}_i{}^j$ are defined by (8.13.10), constitutes a closed system of equations for the functions $\Omega\ (x^i)$, $j^i\ (x^k)$, $M^{ij}\ (x^k)$, $S^i(x^k)$, and $N\ (x^k)$ that is a corollary of spinor equations (8.13.1).

Assuming that $\Omega^2 + N^2 \neq 0$ let us introduce tensors determined by real-valued components ρ, u^i, μ^{ij}, and η:

$$\begin{aligned} &\rho u^i = j^i,\ \rho\mu^{ij} = \Omega M^{ij} + (1/2)\ \varepsilon^{ijks} N M_{ks}, \\ &\Omega + iN = \rho \exp{(i\eta)}. \end{aligned} \tag{8.13.11}$$

If we multiply the first equation in (8.13.10) by Ω/ρ and the second by N/ρ and add the products, we obtain an expression for $\overset{*}{P}_i{}^j$ in terms of the tensor components ρ, u^i, μ^{ij}, S^i, and η:

$$\begin{aligned} \overset{*}{P}_i{}^j = {} & \alpha\,[u_s\partial_i\mu^{js} + u^j\partial_s\mu_i{}^s - S^j\partial_i\eta + u^j\,(S_iu^k - u_iS^k)\ \partial_k\eta] \\ & + 2\alpha\,[(\delta_i^j + u_iu^j)\,(\Omega\varkappa + N\overset{*}{\varkappa}) \\ & + (\delta_i^j u^m - \delta_i^m u^j)\,(\rho\varkappa_m - (1/2)\,\varepsilon_{mnks}\mu^{nk}\overset{*}{\varkappa}{}^s) \\ & + (1/2)\,\delta_i^j\varkappa_{ks}M^{ks} + (1/\rho)\,u^j\,(N\varkappa_{ik} + (1/2)\ \varepsilon_{iksm}\Omega\varkappa^{sm})\ S^k]. \end{aligned} \tag{8.13.12}$$

Using this formula, we can transform Eq. (8.13.5) into

$$(\rho/g)\ d\ (\mu^{ij}\rho)/d\tau + \mu^i{}_n\overset{*}{F}{}^{nj} - \mu^j{}_n\overset{*}{F}{}^{ni} = 0. \tag{8.13.13}$$

Here $d/d\tau = cu^i\partial_i$ and c and g are arbitrary nonzero constants introduced for the sake of convenience in applications while the antisymmetric tensor components $\overset{*}{F}{}^{ij} = -\overset{*}{F}{}^{ji}$ are defined thus:

$$\begin{aligned} \overset{*}{F}{}^{ij} = {} & (c/g)\,\{(\delta_m^i + u_mu^i)\,(\delta_n^j + u_nu^j)\,[\varepsilon^{mnks}u_k\,(-2\overset{*}{\varkappa}_s + \partial_s\eta) \\ & + (2/\rho)\,(\Omega\varkappa^{mn} - (1/2)\,\varepsilon^{mnks}N\varkappa_{ks})] - \partial^iu^j + \partial^ju^i\}. \end{aligned} \tag{8.13.14}$$

Contracting Eqs. (8.13.13) with the tensor components $\varepsilon_{mijk}u^k$ with respect to indices i and j, we obtain

$$\rho d\ (S_m/\rho)/d\tau = g\overset{*}{F}_{mj}S^j. \tag{8.13.15}$$

Since contracting Eqs. (8.13.15) with the tensor components $\varepsilon^{ijmk}u_k$ with respect to m yields Eq. (8.13.13), we see that Eqs. (8.13.13) and (8.13.15) are equivalent. Equations (8.13.15) contain no more than two independent equations, since contraction of Eqs. (8.13.15) with vector components j^m and S^m is ensured identically, in view of the definition of the $\overset{*}{F}{}^{ij}$:

$$u^m\,[\rho d\,(S_m/\rho)/d\tau - g\overset{*}{F}_{mj}S^j] \equiv 0,$$

$$S^m\,[\rho d\,(S_m/\rho)/d\tau - g\overset{*}{F}_{mj}S^j] \equiv 0.$$

We write out the closed system of equations for real-valued tensor components defined by a spinor ψ. This system is a corollary of Eqs. (8.13.1):

$$\begin{aligned}
&\partial_i\rho u^i = 0,\\
&\partial_j\overset{*}{P}_i{}^j = 2\alpha\,[\varkappa\partial_i\Omega + \varkappa_j\partial_i j^j + (1/2)\,\varkappa_{sj}\partial_i M^{sj}\\
&+ \overset{*}{\varkappa}_j\partial_i S^j + \overset{*}{\varkappa}\,\partial_i N],\\
&\rho d\,(S_m/\rho)/d\tau = g\overset{*}{F}_{mj}S^j, \qquad\qquad (8.13.16)\\
&(1/2)\,\partial_i S^i - \varkappa N - (1/4)\,\varepsilon^{ijks}\varkappa_{ij}M_{ks} + \overset{*}{\varkappa}\Omega = 0.
\end{aligned}$$

Here the tensor components $\overset{*}{P}_i{}^j$ and $\overset{*}{F}{}^{ij}$ are defined in (8.13.12) and (8.13.14).

8.14 SOME SOLUTIONS OF DIFFERENTIAL EQUATIONS FOR RELATIVISTIC MODELS OF MAGNETIZABLE FLUIDS WITH INTRINSIC ANGULAR MOMENTUM IN AN ELECTROMAGNETIC FIELD[5]

We assume that space-time is four-dimensional and pseudo-Euclidean with a metric of signature +, +, +, —. We assign to it an inertial Cartesian system of coordinate associated with the observer. Let us consider the following system of differential tensor equations:

$$\begin{aligned}
&\partial_j P_i{}^j = 0,\ \partial_i\rho u^i = 0,\\
&\rho d\,(M_i/\rho)/d\tau = g\overset{*}{F}_{ij}M^j,\ \rho T = \partial\Lambda_0/\partial s, \qquad\qquad (8.14.1)\\
&\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0,\ \partial_j H^{ij} = 0,\\
&\rho T\,ds/d\tau = -cu^i\partial_j\tau_k{}^j.
\end{aligned}$$

Here the $P_i{}^j$ are the components of the energy-momentum tensor, ρ the invariant mass density of the fluid, u^i the components of the velocity 4-vector (dimensionless) of individual particles in the fluid, s the specific entropy density, T the scalar temperature (in equilibrium processes), $\tau_i{}^j$ the components of the tensor that determines the viscous and heat-conducting properties of the fluid, g a constant (the gyromagnetic ratio), $d/d\tau = cu^i\partial_i$ the symbol of the derivative with respect to proper time, c the velocity of light in vacuum, F_{ij} the components of the electromagnetic field tensor, $H_{ij} = F_{ij} - 4\pi M_{ij}$ the components of the electromagnetic induction tensor, and M_{ij} the components of the fluid's body magnetization density tensor related to the components M_i of the body magnetization density 4-vector thus:

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[5] A similar theory but in the framework of Newtonian mechanics has been developed by the author in [9].

$$M_{ij} = -\varepsilon_{ijks} u^h M^s, \quad M_i = (1/2)\, \varepsilon_{ijks} u^j M^{ks}, \tag{8.14.2}$$

where ε_{ijks} is the totally antisymmetric symbol in four dimensions.

The quantity Λ_0 in Eqs. (8.14.1) is defined as a differentiable function of parameters ρ, s, M_i, $d_j M_i$, u_i, and $d_j u_i$, where $d_j = (\delta_j^i - u_j u^i)\, \partial_i$ is the symbol of the spatial derivative. The components $P_i{}^j$ of the energy-momentum tensor and the antisymmetric components $\overset{*}{F}_{ij}$ in Eqs. (8.14.1) are given by the relationships

$$\begin{aligned}
P_i{}^j &= P_{i(f)}^{\,j} + (p+e)\, u_i u^j + p\delta_i^j + \frac{1}{g} u^j M_{ik} \frac{du^k}{d\tau} \\
&+ \frac{\partial \Lambda_0}{\partial d_j M_k} d_i M_k + \frac{\partial \Lambda_0}{\partial d_j u_k} d_i u_k - u^j \sigma_{is} \left[\frac{\partial \Lambda_0}{\partial u_s} - \partial_k \left(\sigma_n^k \frac{\partial \Lambda_0}{\partial d_n u_s} \right) \right] \\
&- M_i u^j u_k \left[\frac{\partial \Lambda_0}{\partial M_k} - \partial_s \left(\sigma_n^s \frac{\partial \Lambda_0}{\partial d_n M_k} \right) \right] \\
&- \frac{1}{c} (u^j \sigma_{is} + u_i \sigma_s^j) \left(\frac{\partial \Lambda_0}{\partial d_s M_k} \frac{dM_k}{d\tau} + \frac{\partial \Lambda_0}{\partial d_s u_k} \frac{du_k}{d\tau} \right) - \tau_i{}^j, \\
\overset{*}{F}_{ij} &= \sigma_i^m \sigma_j^n \left\{ F_{mn} + \varepsilon_{mnkp} u^k \left[\frac{\partial \Lambda_0}{\partial M_p} - \partial_q \left(\sigma_s^q \frac{\partial \Lambda_0}{\partial d_s M_p} \right) \right] \right\} \\
&+ \frac{1}{g} \left(u_i \frac{du_j}{d\tau} - u_j \frac{du_i}{d\tau} \right), \quad \sigma_i^m = \delta_i^m + u_i u^m,
\end{aligned} \tag{8.14.3}$$

where for the tensor components[6] $P_{i(f)}^{\,j}$ and the quantities p and e we have

$$P_{i(f)}^{\,j} = \frac{1}{4\pi} \left[F_{in} H^{jn} - \frac{1}{4} \delta_i^j F_{sm} H^{sm} - u^j u_m (H_{in} F^{mn} - H^{mn} F_{in}) \right],$$

$$p = \rho^2 \frac{\partial \Lambda_0/\rho}{\partial \rho} + M_i \left[\frac{\partial \Lambda_0}{\partial M_i} - \partial_s \left(\sigma_n^s \frac{\partial \Lambda_0}{\partial d_n M_i} \right) \right] - \frac{1}{4} M^{ij} F_{ij},$$

$$e = \Lambda_0 - (1/4)\, M^{ij} F_{ij}.$$

The system of equations (9.14.1)-(9.14.3) with appropriate values of Λ_0 and $\tau_i{}^j$ can be used to describe, within the framework of special relativity, viscous magnetizable fluids with intrinsic angular momentum. The intrinsic angular momentum body density tensor for this specific type of models is given by its components K^{ij}, which are proportional to the magnetization tensor components: $gK^{ij} = M^{ij}$.

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[6] The tensor components $P_{i(f)}^{\,j}$ can be assumed to be the components of the energy-momentum tensor of the electromagnetic field in the medium (see [1]).

The system of equations (8.14.1)-(8.14.3) was derived by means of a variational equation in [1], and the function Λ_0 entering these equations constitutes a part of the matter Lagrangian.

The system of equations (8.14.1) contains equations for energy-momentum, the continuity equation for the mass density of the fluid, the angular momentum equation, the temperature equation, the entropy balance equation, and the Maxwell equations for the electromagnetic field. Here the unknown functions are $u^i(x^j)$, $\rho\,(x^i)$, $T\,(x^i)$, $M_i\,(x^j)$, $s\,(x^i)$, and $F_{ij}(x^i)$. The components u^i of the 4-vector velocity and the components M_i of the magnetization 4-vector entering Eqs. (8.14.1) are, by their definition, connected via the algebraic equations

$$u^iM_i = 0, \quad u_iu^i = -1. \tag{8.14.4}$$

To complete the system of equations (8.14.1)-(8.14.3), we must give the tensor components $\tau_i{}^j$ defining the viscosity and the heat capacity of the fluid. For one, in many cases we may put

$$\tau_i{}^j = \mu\,(\sigma_i^k\partial_k u^j + \sigma^{jk}\partial_k u_i) + [\zeta - (2/3)\,\mu]\,\sigma_i^j\partial_k u^k$$

$$-\,(1/c)\,(u_iq^j + u^jq_i),$$

where q^j are the components of heat flux vector, and μ and ζ are the viscosity coefficients.

Let us introduce in space-time a four-component spinor field $\psi\,(x^i)$, which in the observer's Cartesian system of coordinates is given by its components $\psi^A\,(x^i)$, and the conjugate spinor field $\psi^+\,(x^i)$. For the mass density ρ, the components u^j of the velocity 4-vector, the components M_i of the magnetization 4-vector, and the entropy s, by definition, in Eqs. (8.14.1)-(8.14.3) we put

$$\begin{gathered} \rho u^j = i\psi^+\gamma^j\psi, \quad M_i = m\psi^+\overset{*}{\gamma}_i\psi, \\ \rho\exp i\eta\,(s) = \psi^+\psi + i\psi^+\gamma^5\psi. \end{gathered} \tag{8.14.5}$$

Here γ^j are the Dirac matrices, $\overset{*}{\gamma}_i$ and γ^5 the matrices built on the spin-tensor components given in (8.1.3), m a constant, and $\eta\,(s)$ a fixed differentiable function with a nonzero first derivative in the entire range of variation of s.

Definitions (8.14.5) imply that the fluid's density ρ can be expressed in terms of the spinor fields thus:

$$\rho = [(\psi^+\psi)^2 + (\psi^+\gamma^5\psi)^2]^{1/2}. \tag{8.14.6}$$

In view of the definitions (8.14.5) and according to (8.7.4), Eqs. (8.14.4) are simply identities, while the components M_i of the magnetization 4-vector satisfy an additional algebraic equation

$$M_i M^i = \rho^2 m^2. \tag{8.14.7}$$

Parametrization (8.14.5) imposes no other restrictions on ρ, u^i, M_i, and s except (8.14.7) and (8.14.4).

Let us take the system of equations

$$\begin{gathered}\gamma^i \partial_i \psi + (\varkappa I + i\varkappa_i \gamma^i + \overset{*}{\varkappa}_i \overset{*}{\gamma}{}^i + \overset{*}{\varkappa} \gamma^5)\psi = 0,\\ -\partial_i \psi^+ \gamma^i + \psi^+ (\varkappa I + i\varkappa_i \gamma^i + \overset{*}{\varkappa}_i \overset{*}{\gamma}{}^i + \overset{*}{\varkappa} \gamma^5) = 0,\\ \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \quad \partial_j H^{ij} = 0,\end{gathered} \tag{8.14.8}$$

where the second equation is the conjugate of the first, and the real-valued coefficients $\varkappa$, $\varkappa_i$, $\overset{*}{\varkappa}_i$, and $\overset{*}{\varkappa}$ are determined from the relationships[7]

$$\begin{aligned}\varkappa &= \frac{g}{2mc\rho^2}\Big\{-\Omega\left(\rho\frac{\partial\Lambda_0}{\partial\rho} + \frac{1}{2}F_{ij}M^{ij}\right)\\ &+ N\left[\frac{1}{\partial\eta/\partial s}\left(\frac{\partial\Lambda_0}{\partial s} - \rho T\right) + \frac{c}{g}\,\partial_i M^i\right]\Big\},\\ \overset{*}{\varkappa} &= \frac{g}{2mc\rho^2}\Big\{-N\left(\rho\frac{\partial\Lambda_0}{\partial\rho} + \frac{1}{2}F_{ij}M^{ij}\right)\\ &- \Omega\left[\frac{1}{\partial\eta/\partial s}\left(\frac{\partial\Lambda_0}{\partial s} - \rho T\right) + \frac{c}{g}\,\partial_i M^i\right]\Big\},\\ \varkappa_i &= \frac{g}{2mc\rho}\Big\{\rho G_i + \frac{1}{2}\varepsilon_{jksi}F^{jk}M^s\\ &- \sigma_{ij}\Big[\frac{\partial\Lambda_0}{\partial u_j} - \partial_k\left(\sigma_s^k\frac{\partial\Lambda_0}{\partial d_s u_j}\right)\\ &+ u_s\left(\frac{\partial\Lambda_0}{\partial d_s M_k}d^j M_k + \frac{\partial\Lambda_0}{\partial d_s u_k}d^j u_k\right) + \frac{1}{c}\frac{\partial\Lambda_0}{\partial d_j M_k}\frac{dM_k}{d\tau}\\ &+ \frac{1}{c}\frac{\partial\Lambda_0}{\partial d_j u_k}\frac{du_k}{d\tau}\Big]\Big\} + \frac{1}{2mc\rho}\sigma_{ij}\left(c\partial_s M^{js} + M^{js}\frac{du_s}{d\tau}\right),\\ \overset{*}{\varkappa}_i &= \frac{g}{2c}\left[-\frac{1}{2}\varepsilon_{jksi}F^{jk}u^s - \frac{\partial\Lambda_0}{\partial M^i} + \partial_j\left(\sigma_s^j\frac{\partial\Lambda_0}{\partial d_s M^i}\right)\right]\\ &+ \frac{1}{2}\frac{\partial\eta}{\partial s}\partial_i s + \frac{1}{2}\varepsilon_{ijks}u^j\partial^k u^s.\end{aligned} \tag{8.14.9}$$

Here Ω and N are defined in (8.7.2), ρ, u^i, M^i, and s are expressed in terms of the fields ψ and ψ^+ by formulas (8.14.5), $\sigma_{ij} = g_{ij} + u_i u_j$, g_{ij} the covariant components of the metric tensor in space-time, F_{ij} the components of the electromagnetic field tensor, and M_{ij} the magnetization tensor components determined by the magnetization 4-vector components via (8.14.2).

---

[7] All subsequent results do not change if in Eqs. (8.14.8) instead of $\varkappa_i$ we take $\varkappa_i + \partial_i \gamma$, with $\gamma = \gamma(x^i)$ an arbitrary differentiable function of x^j.

To determine the G_i in the $\varkappa_i$, we introduce the following equations linear in G_i:

$$\rho u^j (\partial_j G_i - \partial_i G_j) + \rho T \, \partial_i s + \partial_j \tau_i{}^j = 0. \tag{8.14.10}$$

The physical meaning of Eqs. (8.14.10) and of the G_i is explained in [1].

Hence, Eqs. (8.14.8) and (8.14.9) for given G_i constitute a system of differential equations for determining the spinor field $\psi(x^i)$, temperature $T(x^i)$, and the components $F_{ij}(x^i)$ of the electromagnetic field tensor.

Let us show that all the equations in (8.14.1) for arbitrary G_i, $\tau_i{}^j$, and T satisfying Eqs. (8.14.10) follow from Eqs. (8.14.8) and (8.14.9) if density, entropy, velocity, and magnetization of the fluid are determined in terms of the fields ψ and ψ^+ via (8.14.5). To this end we consider a system of real-valued tensor equations (8.13.16) at $\varkappa_{ij} = 0$, which are valid due to the spinor equations (8.14.8) and (8.14.9).

As we can easily see, the fourth equation in (8.13.16) with the coefficients $\varkappa$ and $\overset{*}{\varkappa}$ defined by (8.14.9) can be written as

$$\begin{aligned}
&\frac{1}{2}\,\partial_i S^i - \varkappa N + \overset{*}{\varkappa}\Omega = \frac{1}{2m}\,\partial_i M^i \\
&+ \frac{g}{2mc}\left[\frac{1}{\partial\eta/\partial s}\left(-\frac{\partial\Lambda_0}{\partial s} + \rho T\right) - \frac{c}{g}\,\partial_i M^i\right] \\
&= \frac{g}{2mc\,\partial\eta/\partial s}\left(-\frac{\partial\Lambda_0}{\partial s} + \rho T\right) = 0.
\end{aligned}$$

This means that the temperature equation in (8.14.1) is a corollary of Eqs. (8.14.8) and (8.14.9).

According to the definitions (8.7.2) and (8.14.5), the first equation in (8.13.16) is

$$\partial_i j^i = \partial_i \rho u^i = 0.$$

This means that the continuity equation in (8.14.1) is also a corollary of Eqs. (8.14.8) and (8.14.9).

We now calculate the tensor components $\overset{*}{F}_{ij}$ entering the third equation in (8.13.16) and defined in (8.13.14). Substituting $\varkappa_{ij} = 0$ and $\overset{*}{\varkappa}_i$ defined in (8.14.9) into the definition (8.13.14) and doing the necessary transformations, we find that

$$\begin{aligned}
\overset{*}{F}_{ij} &= (c/g)\,[\varepsilon_{ijks} u^k (-2\overset{*}{\varkappa}{}^s + \partial^s\eta) - \partial_i u_j + \partial_j u_i] \\
&= \sigma_i^m \sigma_j^n \left\{F_{mn} + \varepsilon_{mnkp} u^k \left[\frac{\partial\Lambda_0}{\partial M_p} - \partial_q\left(\sigma_s^q \frac{\partial\Lambda_0}{\partial d_s M_p}\right)\right]\right\} \\
&+ (1/g)\,(u_i\, du_j/d\tau - u_j\, du_i/d\tau).
\end{aligned} \tag{8.14.11}$$

The components $\overset{*}{F}_{ij}$ defined in (8.14.11) coincide with those defined in (8.14.3). Hence, the angular momentum equation in (8.14.1) is a corollary of Eqs. (8.14.8) and (8.14.9). It is somewhat more complicated to show [1] that the energy-momentum equation in (8.14.1) is also a corollary of Eqs. (8.14.8) and (8.14.9). We can easily see that the entropy balance equation in (8.14.1) is found by contracting Eq. (8.14.10) with the components u^i of the velocity 4-vector.

Thus, if density, entropy, velocity, and magnetization of a fluid are expressed in terms of spinor fields ψ and ψ^+ via (8.14.5), then, in view of Eqs. (8.14.8)-(8.14.10), all equations (8.14.1) are satisfied identically.

As we can easily see, if the coefficients G_i in (8.14.9) contain derivatives of the determining parameters ρ, u^i, M_i, and s not higher than the second order, the spinor equations in (8.14.8) with the coefficients $\varkappa$, $\varkappa_i$, $\overset{*}{\varkappa}_i$, and $\overset{*}{\varkappa}$ defined in (8.14.9) are second-order differential equations in the components of the spinor ψ and contain components F_{ij} of the electromagnetic field tensor but not their derivatives.

Obviously, if in the system of hydrodynamic equations in (8.14.1) which corresponds to the spinor equations in (8.14.8), we use formulas (8.14.5) for ρ, u^i, M_i, and s, the resulting equations are, in general, third-order differential equations in the components of ψ and first-order differential equations in the F_{ij}. In this sense Eqs. (8.14.8) can be thought of as the integrals of Eqs. (8.14.1)-(8.14.3) with ρ, u^i, M_i, and s expressed in terms of ψ and ψ^+ via (8.14.5). The peculiar feature of these integrals is that for them $M_iM^i = \rho^2m^2$.

In view of the temperature equation $\rho T = \partial\Lambda_0/\partial s$, for the coefficients $\varkappa$ and $\overset{*}{\varkappa}$ defined by (8.14.9) we can write

$$\begin{aligned}\varkappa &= -\frac{g}{2mc\rho^2}\Omega\left(\rho\frac{\partial\Lambda_0}{\partial\rho}+\frac{1}{2}M^{ij}F_{ij}\right)+\frac{N}{2m\rho^2}\partial_iM^i,\\ \overset{*}{\varkappa} &= -\frac{g}{2mc\rho^2}N\left(\rho\frac{\partial\Lambda_0}{\partial\rho}+\frac{1}{2}M^{ij}F_{ij}\right)-\frac{\Omega}{2m\rho^2}\partial_iM^i.\end{aligned} \tag{8.14.12}$$

Direct verification shows that Eqs. (8.14.8), in which the coefficients $\varkappa$ and $\overset{*}{\varkappa}$ are defined by (8.14.12) and the coefficients $\varkappa_i$ and $\overset{*}{\varkappa}_i$ by (8.14.9), are related identically:

$$\begin{aligned}&\psi^+\gamma^5\left[\gamma^i\partial_i\psi+(\varkappa I+i\varkappa_i\gamma^i+\overset{*}{\varkappa}_i\overset{*}{\gamma}^i+\overset{*}{\varkappa}\gamma^5)\psi\right]\\ &\equiv-\left[-\partial_i\psi^+\gamma^i+\psi^+(\varkappa I+i\varkappa_i\gamma^i+\overset{*}{\varkappa}_i\overset{*}{\gamma}^i+\overset{*}{\varkappa}\gamma^5)\right]\gamma^5\psi,\end{aligned}$$

and, hence, contain no more than seven independent real-valued equations. Clearly, if the coefficients $\varkappa$ and $\overset{*}{\varkappa}$ are defined by (8.14.12) and the coefficients $\varkappa_i$ and $\overset{*}{\varkappa}_i$ by (8.14.9), the following system of equations can be taken as a complete system:

$$\begin{aligned}
&\gamma^i\partial_i\psi + (\varkappa I + i\varkappa_i\gamma^i + \overset{*}{\varkappa}_i\overset{*}{\gamma}{}^i + \overset{*}{\varkappa}\gamma^5)\,\psi = 0,\\
&-\partial_i\psi^+\gamma^i + \psi^+(\varkappa I + i\varkappa_i\gamma^i + \overset{*}{\varkappa}_i\overset{*}{\gamma}{}^i + \overset{*}{\varkappa}\gamma^5) = 0,\\
&\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0,\ \partial_j H^{ij} = 0,\ \rho T = \partial\Lambda_0/\partial s.
\end{aligned} \tag{8.14.13}$$

Sometimes it is useful to write the formulas for the coefficients $\varkappa$, $\varkappa_i$, $\overset{*}{\varkappa}_i$, and $\overset{*}{\varkappa}$ in Eqs. (8.14.13) in another form:

$$\begin{aligned}
\varkappa &= \frac{g}{2mc\rho}\left(-\Omega\frac{\partial\Lambda_0}{\partial\rho} + \frac{c}{g\rho}N\,\partial_i M^i\right),\\
\overset{*}{\varkappa} &= \frac{g}{2mc\rho}\left(-N\frac{\partial\Lambda_0}{\partial\rho} - \frac{c}{g\rho}\Omega\partial_i M^i\right),\\
\overset{*}{\varkappa}_i &= \frac{g}{2c}\left[B_i - \frac{\partial\Lambda_0}{\partial M^i} + \partial_j\left(\sigma_s^j\frac{\partial\Lambda_0}{\partial d_s M^i}\right)\right] + \frac{1}{2}\partial_i\eta + \frac{1}{2}\varepsilon_{ijks}u^j\partial^k u^s\\
\varkappa_i &= \frac{g}{2mc\rho}\left\{\rho G_i - M_{ij}E^j - \sigma_{ij}\left[\frac{\partial\Lambda_0}{\partial u_j} - \partial_k\left(\sigma_s^k\frac{\partial\Lambda_0}{\partial d_s u_j}\right)\right.\right.\\
&+ u_s\left(\frac{\partial\Lambda_0}{\partial d_s M_k}d^j M_k + \frac{\partial\Lambda_0}{\partial d_s u_k}d^j u^k\right)\\
&\left.\left.+ \frac{1}{c}\frac{\partial\Lambda_0}{\partial d_j M_s}\frac{dM_s}{d\tau} + \frac{1}{c}\frac{\partial\Lambda_0}{\partial d_j u_s}\frac{du_s}{d\tau}\right]\right\}\\
&+ \frac{1}{2mc\rho}\sigma_{ij}\left(c\partial_s M^{js} + M^{js}\frac{du_s}{d\tau}\right).
\end{aligned} \tag{8.14.14}$$

Here the components E^j of the electric strength 4-vector and the components B_i of the magnetic induction 4-vector are defined thus:

$$E^j = u_m F^{jm},\quad B_i = (1/2)\,\varepsilon_{ijks}u^j F^{ks}.$$

Equations (8.14.8) with coefficients (8.14.9) and Eqs. (8.14.13) with coefficients (8.14.14) are equivalent.

The reader must note that the expressions for the coefficients $\varkappa_i$ and $\overset{*}{\varkappa}_i$ simplify somewhat if we put

$$\Lambda_0 = (2/g)\,M^i\omega_i + \Lambda_m(\rho, s, M_i, d_j M_i, u_i, d_j u_i), \tag{8.14.15}$$

where ω_i are the components of the vorticity 4-vector:

$$\omega_i = (c/2)\,\varepsilon_{ijks}u^j\partial^k u^s.$$

In this case the coefficients $\varkappa$, $\varkappa_i$, $\overset{*}{\varkappa}_i$, and $\overset{*}{\varkappa}$ in Eqs. (8.14.13) take the following simpler form:

$$\varkappa = \frac{g}{2mc\rho}\left(-\Omega\frac{\partial\Lambda_m}{\partial\rho} + \frac{c}{g\rho}N\partial_i M^i\right),$$

$$\overset{*}{\varkappa} = \frac{g}{2mc\rho}\left(-N\frac{\partial\Lambda_m}{\partial\rho} - \frac{c}{g\rho}\Omega\partial_i M^i\right),$$

$$\overset{*}{\varkappa}_i = \frac{g}{2c}\left[B_i - \frac{\partial\Lambda_m}{\partial M^i} + \partial_j\left(\sigma_s^j\frac{\partial\Lambda_m}{\partial d_s M^i}\right)\right] + \frac{1}{2}\partial_i\eta,$$

$$\varkappa_i = \frac{g}{2mc\rho}\left\{\rho G_i - M_{ij}E^j - \sigma_{ij}\left[\frac{\partial\Lambda_m}{\partial u_j} - \partial_k\left(\sigma_s^k\frac{\partial\Lambda_m}{\partial d_s u_j}\right)\right]\right.$$
$$+ u_s\left(\frac{\partial\Lambda_m}{\partial d_s M_k}d^j M_k + \frac{\partial\Lambda_m}{\partial d_s u_k}d^j u_k\right) + \frac{1}{c}\frac{\partial\Lambda_m}{\partial d_j M_s}\frac{dM_s}{d\tau}$$
$$\left.+\frac{1}{c}\frac{\partial\Lambda_m}{\partial d_j u_s}\frac{du_s}{d\tau}\right]\Bigg\}.$$

If the G_i that enter $\varkappa_i$ and satisfy Eqs. (8.14.10) are given as functions of the main parameters of fluid and field, the system of equations (8.14.8) and (8.14.9) is complete. Defining the G_i is closely linked with the concrete thermodynamic and mechanical formulations of the problem.

We note that if we put

$$T = d\lambda/d\tau;\quad G_i = -c\lambda\partial_i s,\quad \tau_i^j = 0,$$

where $\lambda = \lambda(x^i)$ is an arbitrary differentiable function, and c the speed of light in vacuum, Eq. (8.14.10) is satisfied identically if $ds/d\tau = 0$. This case corresponds to adiabatic processes in a magnetizable liquid.

Equation (8.14.10) is also satisfied identically if we put $s = \text{const}$, $\tau_i^j = 0$, and $G_i = 0$. This case corresponds to isentropic processes in an ideal magnetizable liquid.

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